

$C^{\sigma+\alpha}$ REGULARITY FOR CONCAVE NONLOCAL FULLY NONLINEAR ELLIPTIC EQUATIONS WITH ROUGH KERNELS

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ABSTRACT. We establish $C^{\sigma+\alpha}$ interior estimates for concave nonlocal fully nonlinear equations of order $\sigma \in (0, 2)$ with rough kernels. Namely, we prove that if $u \in C^\alpha(\mathbb{R}^n)$ solves in B_1 a concave translation invariant equation with kernels in $\mathcal{L}_0(\sigma)$, then u belongs to $C^{\sigma+\alpha}(\overline{B_{1/2}})$, with an estimate. More generally, our results allow the equation to depend on x in a C^α fashion.

Our method of proof combines a Liouville theorem and a blow-up (compactness) procedure. Due to its flexibility, the same method can be useful in different regularity proofs for nonlocal equations.

1. INTRODUCTION AND RESULTS

In the paper [5], Caffarelli and Silvestre established the $C^{\sigma+\alpha}$ interior regularity for concave translation invariant nonlocal fully nonlinear equations of order $\sigma \in (0, 2)$ with smooth kernels. This result extended the classical $C^{2,\alpha}$ interior estimates for concave second order elliptic equations of Evans [6] and Krylov [9] to the context of integro-differential equations. The main result in [5] states that if $u \in L^\infty(\mathbb{R}^n)$ satisfies $\inf_a L_a u = 0$ in B_1 and $L_a \in \mathcal{L}_2(\sigma)$ for all a , then $u \in C^{\sigma+\alpha}(\overline{B_{1/2}})$, with an estimate.

The ellipticity class $\mathcal{L}_2 = \mathcal{L}_2(\sigma)$ is defined as the set of all linear translation invariant operators of the form

$$L_a u = \int_{\mathbb{R}^n} \frac{1}{2} (u(x+y) + u(x-y) - 2u(x)) K_a(y) dy, \quad (1.1)$$

where K_a are even kernels satisfying

$$0 < \frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} \leq K_a(y) \leq \frac{\Lambda(2-\sigma)}{|y|^{n+\sigma}} \quad (1.2)$$

and, in addition, with all its second order partial derivatives satisfying the following scaling invariant bounds away from the origin:

$$[K_a]_{C^2(\mathbb{R}^n \setminus B_\rho)} \leq \Lambda(2-\sigma) \rho^{-n-\sigma-2} \quad \text{for all } \rho > 0. \quad (1.3)$$

The class \mathcal{L}_2 is a subclass of the class \mathcal{L}_0 , where \mathcal{L}_0 is formed by all operators of the form (1.1) that satisfy (1.2) but not necessarily (1.3). The bounds by above and by below in (1.2) allow the kernels in \mathcal{L}_0 to be very oscillating and irregular, and that is why they are referred to as *rough kernels*.

After the paper [5], the following two main questions in the regularity theory of concave nonlocal fully nonlinear elliptic equations remained open.

A first open question was to determine whether the same $C^{\sigma+\alpha}$ estimates held also for non-smooth kernels. In this direction, to prove the interior $C^{\sigma+\alpha}$ regularity for the equation

$$M_{\mathcal{L}_0}^- u = 0 \quad \text{in } B_1. \quad (1.4)$$

is the fifth open problem listed in the wiki of Nonlocal Equations [15]. Recall that the extremal operator for the class \mathcal{L}_0 is defined as

$$M_{\mathcal{L}_0}^- u(x) = \inf_{L \in \mathcal{L}_0} Lu(x) = \int_{\mathbb{R}^n} \left\{ \lambda(\delta^2 u(x, y))^+ - \Lambda(\delta^2 u(x, y))^- \right\} \frac{2 - \sigma}{|y|^{n+\sigma}} dy. \quad (1.5)$$

Here, and throughout the article, we use the following notation for second order incremental quotients

$$\delta^2 u(x, y) = \frac{1}{2}(u(x + y) + u(x - y) - 2u(x)).$$

The equation (1.4) is arguably the canonical example of concave equation of order σ . As given in (1.5), $M_{\mathcal{L}_0}^-$ has a simple “closed expression”, similar to

$$M^-(D^2 u) = \lambda \text{tr}(D^2 u)^+ - \Lambda \text{tr}(D^2 u)^-$$

for the second order Pucci. Such a closed expression is not available for $M_{\mathcal{L}_2}^-$. However, the equation (1.4) is not covered by the theory in [5] since it is elliptic with respect to \mathcal{L}_0 but not with respect to \mathcal{L}_2 .

A second natural question that remained open after the paper [5] was to prove a $C^{\sigma+\alpha}$ Schauder type estimate for non translation invariant equations with C^α dependence on x . In the case of second order fully nonlinear elliptic equations, this Schauder estimate is a classical result. It is proved by exploiting the fact that in a small neighborhood of a given point the equation is a small perturbation of a translation invariant equation —this is the nonlinear perturbation method of Caffarelli [1]. To prove $C^{\sigma+\alpha}$ regularity for equations of order $\sigma \in (0, 2)$, the same method does not work essentially because if u is a function with a zero of order $\sigma + \alpha$ at $x = 0$, the scaling $\rho^{-\sigma-\alpha} u(\rho \cdot)$, $\rho \ll 1$ typically results in a growth of the type $|x|^{\sigma+\alpha}$ at infinity, which is not integrable against the tails of the kernel. This difficulty will be described in more detail later on in the introduction.

In this paper we answer the previous two questions. More precisely, we establish existence, uniqueness, and $C^{\sigma+\alpha}$ interior regularity, for nonlocal Dirichlet problems of the form

$$\begin{cases} I(u, x) = 0 & \text{in } B_1 \\ u = g & \text{in } \mathbb{R}^n \setminus B_1, \end{cases} \quad (1.6)$$

where I is a concave operator, elliptic with respect to \mathcal{L}_0 , and depending on x in C^α fashion —see assumptions (1.7)-(1.8)-(1.9)-(1.10) below. We prove that if $g \in C^\alpha(\mathbb{R}^n \setminus B_1)$ (with α small), then there exists a unique viscosity solution to the problem (1.6), which is $C^{\sigma+\alpha}$ in the interior of B_1 —with an estimate in $\overline{B_{1/2}}$.

For equations with rough kernels, our assumption on the *complement data* (or *exterior data*) $g \in C^\alpha$ can not be weakened to $g \in L^\infty$ —as in [5]. Indeed, in the paper we find a sequence of functions $u_m \in C(\mathbb{R}^n)$ that solve in the viscosity sense $M_{\mathcal{L}_0}^+ u_m = 0$ in B_1 and satisfy $\|u_m\|_{L^\infty(\mathbb{R}^n)} = 1$ but $\|u_m\|_{C^{\sigma+\alpha}(\mathbb{R}^n)} \nearrow +\infty$ as $m \rightarrow \infty$ for all $\alpha > 0$. Hence, a $C^{\sigma+\alpha}$ interior estimate can not hold for any $\alpha > 0$ with merely bounded complement data. To construct such sequence u_m we exploit the strong sensitivity of nonlocal operators with rough kernels to quickly oscillating complement data, to the point that interior regularity can be broken “from the exterior” by choosing very oscillating exterior data. However, the more regular the tails of the kernel are, the less sensitive to far oscillations. In this direction, we prove that when the kernels belong to the class \mathcal{L}_α —a scaling invariant class of C^α kernels— solutions to concave equations with merely bounded complement data do have $C^{\sigma+\alpha}$ interior regularity.

A main difficulty of nonlocal operators with rough kernels is that, as said above, they are very sensitive to oscillations in the complement data. This does not happen with smooth kernels because high frequency oscillations balance out when they are integrated against a kernel with smooth tails. This idea is recurrently exploited in the proofs of [5], essentially by transferring derivatives from the function to the (smooth) kernels with a sort of integration by parts. Since we can not do the same with rough kernels, we need a different approach.

Similarly as in the concave case, the $C^{1+\gamma}$ regularity for general nonlocal fully nonlinear equations was first established for smooth kernels, and only posteriorly extended to rough kernels. In [3], Caffarelli and Silvestre obtained $C^{1+\gamma}$ interior estimates for these equations in the intermediate class of kernels \mathcal{L}_1 —those satisfying (1.3) with C^2 replaced by C^1 and $\rho^{-n-\sigma-2}$ replaced by $\rho^{-n-\sigma-1}$. It was Kriventsov [10] to establish the $C^{1+\gamma}$ estimates for elliptic equations of order $\sigma > 1 + \gamma$ with rough kernels, that is, for \mathcal{L}_0 . The proof in [10] combines a new estimate for solutions with Lipschitz complement data and perturbative (compactness) arguments à la [4].

Later, in [13], we gave a new proof of the result in [10], extending it also to the parabolic case. The key idea of this new proof was to deduce the interior regularity from a Liouville theorem, via a blow-up (compactness) argument. In the present paper, we refine and improve significantly this method of proof from [13] in order to obtain the $C^{\sigma+\alpha}$ estimates for concave equations. Moreover, the methods of this paper are flexible enough to be applied in other contexts. For instance, the ideas we introduce here —suitably adapted— are crucial in the paper [11], by Ros-Oton and the author, where the boundary regularity (of order $1 + s + \alpha$) for fully nonlinear elliptic integro-differential equations of order $2s$ is established.

As said above, the results of this paper apply to non translation invariant equations with C^α dependence on x . More precisely, while in [3, 4, 10] the kernels depend only in y —i.e. $K_a = K_a(y)$ as in (1.1)—, here we include kernels $K_a(x, y)$ which are C^α in the variable x (in the appropriate sense) and rough in the variable y .

We consider concave operators of the form

$$I(u, x) := \inf_{a \in \mathcal{A}} \left(\int_{\mathbb{R}^n} \delta^2 u(x, y) K_a(x, y) dy + c_a(x) \right), \quad (1.7)$$

where \mathcal{A} is some index set. We assume that for all $a \in \mathcal{A}$, for all x and x' in \mathbb{R}^n , and for all $r > 0$ we have

$$\frac{\lambda(2 - \sigma)}{|y|^{n+\sigma}} \leq K_a(x, y) \leq \frac{\Lambda(2 - \sigma)}{|y|^{n+\sigma}}, \quad (1.8)$$

$$\int_{B_{2r} \setminus B_r} |K_a(x, y) - K_a(x', y)| dy \leq A_0 |x - x'|^\alpha \frac{2 - \sigma}{r^\sigma}, \quad (1.9)$$

and

$$\|c_a\|_{C^\alpha(B_1)} \leq C_0, \quad (1.10)$$

where $\lambda \leq \Lambda$, A_0 and C_0 are given constants.

The following is the main result of the paper.

Theorem 1.1. *Let $\sigma \in (0, 2)$, and λ , Λ , A_0 , and C_0 be given constants with $0 < \lambda \leq \Lambda$. Then, there exists $\bar{\alpha} > 0$ depending only on n , σ , λ , Λ such that the following statement holds.*

Let $\alpha \in (0, \bar{\alpha})$ such that $\sigma + \alpha$ is not an integer. Assume that $u \in C^{\sigma+\alpha}(B_1) \cap C^\alpha(\mathbb{R}^n)$ is a solution of

$$I(u, x) = 0 \quad \text{in } B_1,$$

where I is of the form (1.7) and satisfying (1.8), (1.9), and (1.10). We then have

$$\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|u\|_{C^\alpha(\mathbb{R}^n)}),$$

where C_0 is the constant from (1.10) and where C depends only on n , σ , α , λ , Λ , and A_0 .

Some comments are in order.

- Theorem 1.1 is stated as an *a priori* estimate: we assume that $u \in C^{\sigma+\alpha}(B_1)$ (with no quantitative control on the norm) and we obtain a $C^{\sigma+\alpha}$ estimate in $\overline{B_{1/2}}$. From this *a priori* estimate, by using the regularization procedure of Section 4, we will deduce the existence and uniqueness of a (classical) $C^{\sigma+\alpha}$ solution to the convex equation $I(u, x) = 0$ in B_1 with given C^α exterior data —see Theorem 1.3.
- As said above, the estimate of Theorem 1.1 would be false if we replaced $\|u\|_{C^\alpha(\mathbb{R}^n)}$ in its right hand side by $\|u\|_{L^\infty(\mathbb{R}^n)}$ —see Section 5.
- With minor changes in the proofs we can show the dependence of C only on a lower bounds for σ and for the gap between $\sigma + \alpha$ and its integer part. To do it, we can modify the proof of Proposition 3.1, adding an additional sequence of orders $\sigma_k \in [\sigma_0, 2]$ as in [13]. Everything in the paper is prepared so that this can be done (notice in particular that in the statement of the Liouville theorem in Section 3, the exponent $\bar{\alpha}$ does not depend on σ). However, since the proof of Proposition 3.1 is already quite involved as it is, we have chosen

not to do this, not to distract the attention from the real difficulties of the problem.

The following corollary provides with a $C^{\sigma+\alpha}$ interior estimate for solutions u that are merely bounded in \mathbb{R}^n when the kernels are C^α —recall that for rough kernels this is not possible. We introduce the class \mathcal{L}_α of operators of the form (1.1) with kernels K_a satisfying (1.2) and (1.3) with C^2 replaced by C^α and $\rho^{-n-\sigma-2}$ replaced by $\rho^{-n-\sigma-\alpha}$ —note that this is consistent with the definition of \mathcal{L}_2 and \mathcal{L}_1 . In the case of non translation invariant operators we will require the following regularity condition in the variable y :

$$[K_a(x, \cdot)]_{C^\alpha(\mathbb{R}^n \setminus B_\rho)} \leq \Lambda(2 - \sigma)\rho^{-n-\sigma-\alpha} \quad \text{for all } \rho > 0. \quad (1.11)$$

Corollary 1.2. *Let $\sigma, \lambda, \Lambda, A_0, C_0$, and $\bar{\alpha}$ as in Theorem 1.1.*

Let $\alpha \in (0, \bar{\alpha})$ such that $\sigma + \alpha$ is not an integer. Assume that $u \in C^{\sigma+\alpha}(B_1) \cap L^\infty(\mathbb{R}^n)$ is a solution of

$$I(u, x) = 0 \quad \text{in } B_1,$$

with I is defined by (1.7) and satisfying (1.8), (1.9), (1.10), and (1.11). Then,

$$\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}),$$

where C_0 is the constant from (1.10) and C depends only on $n, \sigma, \alpha, \lambda, \Lambda$, and A_0 .

In order to give an existence and uniqueness result for non translation invariant equations, we need to introduce a regularization procedure based in the one from [5]. We find regularized equations $I^\epsilon(u^\epsilon, x) = 0$ that have C^3 solutions and that converge to $I(u, x) = 0$ as $\epsilon \searrow 0$ (in the appropriate sense). A novelty with respect to [5] is that for our non translation invariant equations we do not have a comparison principle between viscosity solutions. Hence, our set up of Perron's method can not rely in the viscosity comparison principle but rather in a property of “classical solvability in tiny balls” for the regularized equations.

Using the regularization procedure and the a priori estimates of Theorem 1.1 and Corollary 1.2 we can prove the following existence and uniqueness result.

Theorem 1.3. *Let $\sigma, \lambda, \Lambda, A_0, C_0, \bar{\alpha}, \alpha$ as in Theorem 1.1.*

Consider the nonlinear Dirichlet problem (1.6), where I , defined by (1.7), satisfies (1.8), (1.9), and (1.10), and where g is a bounded function belonging to $C(\mathbb{R}^n)$. Assume that either

$$(a) \ g \in C^\alpha(\mathbb{R}^n \setminus B_1)$$

or

$$(b) \ I \text{ satisfies (1.11).}$$

Then, there exists a classical solution $u \in C^{\sigma+\alpha}(B_1) \cap C(\mathbb{R}^n)$ of (1.6). As a consequence, the solution u is the unique viscosity solution to (1.6).

Moreover, this solution u satisfies, in case (a), the estimate

$$\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)}),$$

and, in case (b), the estimate

$$\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{L^\infty(\mathbb{R}^n)}),$$

where C_0 is the constant from (1.10) and C depends only on n , σ , α , λ , Λ , and A_0 .

A key idea in our proofs is to deduce the interior regularity results from a Liouville theorem, by using a blow-up (compactness) argument. As a general type of proof in PDEs, proving regularity from a Liouville theorem is a well-known strategy that has been used in a large variety of problems. However, to our knowledge it had not been applied to fully nonlinear elliptic equations until recently by the author in [13] —a reason explaining this may be that for second order equations these type of argument gives nothing new with respect to classical perturbative methods.

In the context of nonlocal equations, this method has two main advantages. First, the Liouville theorem approach allows us to work with solutions in the whole space —rather than only in B_1 , say. This makes possible to deal with rough kernels: we are not troubled the sensitivity to the exterior data because “there is no exterior data”. Second, since we blow up the equation, we typically retrieve a translation invariant limiting equation even in when the original equation is not. This is what allows us to obtain $C^{\sigma+\alpha}$ Schauder estimates for non translation invariant equations. Similarly, we could deal with certain “lower order terms” which disappear after blow-up. For instance, our proof immediately applies to the case of truncated kernels.

The outlines of our strategy of proof are the following. First, we prove a Liouville theorem for global solutions satisfying a certain geometric growth control at infinity. To do it, we essentially apply a regularity proof to a global solution. Using the scaling of the equation and the growth control, we obtain seminorm estimates in every ball B_R that, letting $R \rightarrow \infty$, imply that the global solution is a polynomial. Second, with the new Liouville theorem at hand, we use a blow up contradiction argument to deduce a interior regularity estimate for solutions *only* in B_1 .

The faster is the growth allowed in the Liouville theorem, the better the regularity we will prove with it. For instance, to prove $C^{1+\gamma}$ regularity for fully nonlinear elliptic equations of order $\sigma > 1 + \gamma$ the required Liouville theorem states: “if u is a global solution and $|u(x)| \leq 1 + |x|^{1+\gamma}$ for all $x \in \mathbb{R}^n$, then u is affine”. This Liouville theorem is quite easy to prove using the Hölder estimates from [3].

Similarly, to obtain $C^{\sigma+\alpha}$ estimates we will need a theorem stating: “if u is a global solution to a concave equation and $|u(x)| \leq 1 + |x|^{\sigma+\alpha}$ for all $x \in \mathbb{R}^n$ then u is a polynomial of degree two” (here we are thinking on the most delicate case $\sigma + \alpha > 2$). The problem now is that it is not so clear how to translate this informal statement into a rigorous one. The most evident difficulty is that for functions growing at infinity like $|x|^{\sigma+\alpha}$ our equation of order σ is meaningless, since the operators cannot be computed at such functions. They grow too fast and they are not integrable against the tails of the kernel decaying like $|y|^{-n-\sigma}$.

An important point in the paper is to find an appropriate statement for this Liouville theorem, which is given in Theorem 2.1 in Section 3. Since the equation is

meaningless due to the fast growth, Theorem 2.1 is not stated for viscosity solutions to some equation but rather for functions satisfying the three conditions (i)-(iii) in its statement. Unlike the equation, these three conditions make sense under the growth $|x|^{\sigma+\alpha}$ and they summarize the relevant information of “being solution” to some concave fully nonlinear equation.

As said above, the ideas of the present paper are quite flexible and can be applied to different situations. In the beginning of this introduction we have referred to the application to boundary regularity by Ros-Oton and the author [11]. In more detail, the main result in [11] states that if $u \in L^\infty(\mathbb{R}^n)$ is a solution of $Iu = 0$ in B_1^+ and $u = 0$ in B_1^- , with I elliptic with respect to the class of homogeneous kernels

$$\left\{ \frac{a(y/|y|)}{|y|^{n+2s}}, a(y) = a(-y), \lambda \leq a \leq \Lambda, \|a\|_{C^{1+\alpha-s}(S^{n-1})} \leq \Lambda \right\},$$

then

$$u(x) - (p \cdot x + b)(x_n)_+^s = o(|x|^{1+s+\alpha}) \quad \text{in } B_1^+ \text{ for } x \sim 0,$$

for some $p \in \mathbb{R}^n$ and $b \in \mathbb{R}$ bounded and for some $\alpha > 0$ small. This result contains in the limit $s \nearrow 1$ the classical boundary regularity theory of Krylov for fully nonlinear elliptic equation of second order. Following the method from [13] and the present paper, the result of [11] is obtained using blow-up and compactness from a Liouville theorem —which, in the case of boundary regularity is for solutions in \mathbb{R}_+^n growing less than $|x|^{1+s+\alpha}$ as $x \rightarrow \infty$. Since $1 + s + \alpha$ exceeds $2s$ some of the difficulties that we meet in the boundary regularity result are similar to the ones of this paper, and we can solve them by suitably adapting the ideas of this paper to the boundary regularity context.

Another result in which the methods of the present paper have been very useful is the linear regularity theory for the infinitesimal generator of a general symmetric stable Lévy process, also by Ros-Oton and the author [12]. A main result in [12] is a interior estimate for all equations of the form

$$(1-s) \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \frac{d\mu(y/|y|)}{|y|^{n+2s}} = f(x) \quad \text{in } B_1,$$

where $f \in C^\alpha(B_1)$ and where μ is a probability measure on S^{n-1} . Solutions $u \in C^\alpha(\mathbb{R}^n)$ to the previous equation are shown to belong to $C^{2s+\alpha}(B_{1/2})$ provided that the measure μ is *not* supported on some hyperplane (intersected with S^{n-1}). Clearly this is also a necessary condition for regularity, because when μ is supported on some hyperplane the equation will not regularize in the direction orthogonal to the hyperplane.

After finishing a previous version (published online as a preprint) of this paper, Luis Silvestre let us know about the preprint of Tianling Jin and Jingang Xiong [8], where they prove Schauder estimates ($C^{\sigma+\alpha}$) for $L^\infty(\mathbb{R}^n)$ solutions to concave fully nonlinear equations with smooth kernels in \mathcal{L}_2 and C^α dependence on x . Our Corollary 1.2 applies in particular to this situation since $\mathcal{L}_\alpha \subset \mathcal{L}_2$. Their results and ours are independent, with different proofs, and both preprints were published

online the same day. As explained in this introduction, Schauder estimates for non translation invariant equations were a main open issue in nonlocal equations and thus believe that it is of interest to have now two different proofs of these estimates in the case of smooth kernels.

The paper is organized as follows. In Section 2 we state and prove the Liouville theorem that serves to obtain $C^{\sigma+\alpha}$ regularity. In Section 3 we state and prove Proposition 3.1 (containing the compactness argument) and use it to prove Theorems 1.1 and Corollary 1.2. The regularization procedure and the proof of Theorem 1.3 are given in Section 4. Finally, in Section 5 we give the counterexamples to $C^{\sigma+\alpha}$ interior regularity under the mere assumption of bounded complement data.

Throughout the paper we will use the following conventions:

- Given $\beta > 0$ which is not an integer we will denote as C^β the space $C^{k,\beta'}$ where $k = \lfloor \beta \rfloor$ is the floor of β and $\beta' = \beta - k$.
- The square brackets $[\cdot]$ will stand for seminorms. For example, when $\sigma + \alpha \in (2, 3)$, $[u]_{C^{\sigma+\alpha}(B_1)}$ will denote the $C^{\sigma+\alpha-2}$ Hölder seminorm of D^2u .
- The constants λ and Λ are sometimes referred to as the “ellipticity constants”.

2. THE LIOUVILLE THEOREM

In this section we state and prove the Liouville theorem that serves to prove $C^{\sigma+\alpha}$ interior regularity.

Theorem 2.1. *Let $\sigma_0 \in (0, 2)$ and $\sigma \in [\sigma_0, 2)$. There is $\bar{\alpha} > 0$ depending only on n , σ_0 , and ellipticity constants such that the following statement holds.*

Let α and α' be constants satisfying $0 < \alpha' < \alpha < \bar{\alpha}$. Assume that $u \in C_{\text{loc}}^{\sigma+\alpha'}(\mathbb{R}^n)$ satisfies the following properties.

- (i) *There exists $C_1 > 0$ such that for all $\beta \in [0, \sigma + \alpha']$ and for all $R \geq 1$ we have*

$$[u]_{C^\beta(B_R)} \leq C_1 R^{\sigma+\alpha-\beta}.$$

- (ii) *For all $h \in \mathbb{R}^n$ we have*

$$M_{\mathcal{L}_0}^-(u(\cdot + h) - u) \leq 0 \leq M_{\mathcal{L}_0}^+(u(\cdot + h) - u) \quad \text{in } \mathbb{R}^n.$$

- (iii) *For every nonnegative $\mu \in L^1(\mathbb{R}^n)$ with compact support and $\int_{\mathbb{R}^n} \mu(h) dh = 1$, we have*

$$M_{\mathcal{L}_0}^+ \left(\oint u(\cdot + h) \mu(h) dh - u \right) \geq 0 \quad \text{in } \mathbb{R}^n.$$

Then, $u(x)$ is a polynomial of degree ν , where ν is the floor (or integer part) of $\sigma + \alpha$.

In (iii), and in the rest of the paper, the symbol \oint means average (integral with respect to the measure total mass one $\mu(h) dh$). We write \oint even if we could equivalently write \int as a reminder of the assumption $\int_{\mathbb{R}^n} \mu(h) dh = 1$.

Throughout the paper, α' will be a constant in $(0, \alpha)$. We will sometimes require, in addition, that $\nu < \sigma + \alpha'$ where $\nu = \lfloor \sigma + \alpha \rfloor$. An α' satisfying both conditions exists when $\sigma + \alpha$ is not an integer. In all the paper, one can think of α' as given explicitly by

$$\alpha' := \max \left\{ \frac{\alpha}{2}, \frac{\sigma + \alpha + \nu}{2} - \sigma \right\}. \quad (2.1)$$

The statement of Theorem 2.1 requires some more detailed explanation. Note that the L^∞ growth condition in (i) $\|u\|_{L^\infty(B_R)} \leq CR^{\sigma+\alpha}$ is too loose for $M_{\mathcal{L}_0}^+ u(x)$ and $M_{\mathcal{L}_0}^- u(x)$ to be defined, even though $u \in C^{\sigma+\alpha'}$.

However, the control in (i) implies that for every $\beta \in (0, \min\{1, \sigma + \alpha'\})$ we have

$$\|u(\cdot + h) - u\|_{L^\infty(B_R)} \leq C|h|^\beta R^{\alpha+\sigma-\beta}. \quad (2.2)$$

Therefore, taking $\beta \in (\alpha, \min\{1, \sigma + \alpha'\})$ in (2.2) we find that the $C^{\sigma+\alpha'}$ function $u(\cdot + h) - u$ belongs to $L^1(\mathbb{R}^n, \omega_\sigma)$ —here and throughout the paper ω_σ denotes the weight

$$\omega_\sigma(y) = (1 + |y|)^{-n-\sigma}.$$

Thus $M_{\mathcal{L}_0}^+$ and $M_{\mathcal{L}_0}^-$ of $u(\cdot + h) - u$ are well defined pointwise, and the inequalities in (ii) are meaningful in the classical sense.

Likewise, the function $\int u(\cdot + h)\mu(h)dh - u$ is $C^{\sigma+\alpha'}$ and belongs to $L^1(\mathbb{R}^n, \omega_\sigma)$ —recall that when μ has compact support. Thus, the inequality in assumption (iii) is—also in this case—meaningful in the classical sense.

Remark 2.2. When $\sigma \leq 1$ the proof of this Liouville theorem simplifies significantly and the assumption (iii) is not needed. In this case, the theorem follows from iterating the $C^\gamma(\overline{B_{1/2}})$ estimate in [3, Theorem 12.1] for solutions $v \in L^\infty(B_1) \cap L^1(\mathbb{R}^n, \omega_\sigma)$ to the two viscosity inequalities $M_{\mathcal{L}_0}^- v \leq 0 \leq M_{\mathcal{L}_0}^+ v$ in B_1). Applying this C^γ estimate to incremental quotients of u at every scale and iterating (like in the proof of $C^{1+\gamma}$ regularity for fully nonlinear equations) we obtain

$$[u]_{C^{1+\gamma}(B_R)} \leq CR^{\alpha+\sigma-1-\gamma}.$$

Then, since $\sigma \leq 1$ the conclusion of the theorem holds taking $R \nearrow \infty$ provided that $\alpha < \gamma$. For a very similar argument in the parabolic setting see [13].

Proof of Theorem 2.1. The result for all $C_1 > 0$ trivially follows from the result for $C_1 = 1$. Thus, in all the proof we assume that $C_1 = 1$.

In this proof we follow to a large extent the exposition in the lecture notes of Silvestre [14], where an insightful sketchy version of the $C^{\sigma+\alpha}$ regularity proof from [5] is given. In the present Liouville theorem setting, however, the same “simplified” argument (with few modifications) provides with a short complete proof. This is because since the equation holds in all the space there is no need to truncate functions, avoiding many technical complications.

We want to prove that for some $\bar{\alpha}$ depending only on n, σ_0, λ , and Λ (but *not* on α' nor α) we have

$$[u]_{C^{\sigma+\bar{\alpha}}(B_R)} \leq CR^{\alpha-\bar{\alpha}}, \quad (2.3)$$

with C independent of R . Once this will be proved, since $\alpha < \bar{\alpha}$, sending R to infinity the theorem will follow.

Let us define

$$P(x) := \int_{\mathbb{R}^n} (\delta^2 u(x, y) - \delta^2 u(0, y))^+ \frac{2-\sigma}{|y|^{n+\sigma}} dy$$

and

$$N(x) := \int_{\mathbb{R}^n} (\delta^2 u(x, y) - \delta^2 u(0, y))^- \frac{2-\sigma}{|y|^{n+\sigma}} dy.$$

Using (i) —recall that $C_1 = 1$ — we find that P and N are $C^{\alpha'}$ and satisfy

$$0 \leq P \leq CR^\alpha \quad \text{and} \quad 0 \leq N \leq CR^\alpha \quad \text{in } B_R, \quad (2.4)$$

for all $R \geq 1$, with C universal (meaning that it depends only on n, σ_0, λ , and Λ). Indeed, let us prove (2.4) when $\nu = \lfloor \sigma + \alpha \rfloor = 2$ (the cases $\nu = 0$ and $\nu = 1$ are very similar). Using that $[u]_{C^{\sigma+\alpha'}(B_2)} \leq 1$ and that $[u]_{C^\beta(B_R)} \leq R^{\sigma+\alpha-\beta}$ we obtain, taking $\beta \in (\alpha, \min\{1, \sigma + \alpha'\})$, that

$$|\delta^2 u(x, y) - \delta^2 u(x', y)| \leq \begin{cases} C|y|^2 d^{\sigma+\alpha'-2} & \text{for } y \in B_d \\ Cd^2 |y|^{\sigma+\alpha'-2} & \text{for } y \in B_{1/2} \setminus B_d \\ Cd^\beta R^{\sigma+\alpha-\beta} & \text{for } y \in B_R \setminus B_{1/2}. \end{cases}$$

Therefore,

$$\begin{aligned} |P(x) - P(x')| &\leq \int_{\mathbb{R}^n} |\delta^2 u(x, y) - \delta^2 u(x', y)| \frac{2-\sigma}{|y|^{n+\sigma}} dy \\ &\leq C \int_{\mathbb{R}^n} (|y|^2 d^{\sigma+\alpha'-2} \chi_{B_d}(y) + d^2 |y|^{\sigma+\alpha'-2} \chi_{B_{1/2} \setminus B_d}(y) + \\ &\quad + d^\beta |y|^{\sigma+\alpha-\beta} \chi_{\mathbb{R}^n \setminus B_{1/2}}(y)) \frac{2-\sigma}{|y|^{n+\sigma}} dy \\ &\leq C(d^{\alpha'} + d^\beta) \leq Cd^{\alpha'}. \end{aligned} \quad (2.5)$$

This shows that $P \in C^{\alpha'}(B_1)$. Taking $x' = 0$ in (2.5) we obtain the bound by above for P in B_1 of (2.4). To prove the same bound in B_R for all $R \geq 1$ we use rescaling. Given $\rho > 0$ we consider the rescaled function

$$\bar{u}(x) = \rho^{-\sigma-\alpha} u(\rho x)$$

It is immediate to verify that \bar{u} satisfies the same assumptions (i), (ii), and (iii) as u . In particular the constant C_1 in (i) for \bar{u} is the same as that of u , that is $C_1 = 1$. Then, as we have proved before for u , we have

$$0 \leq \int_{\mathbb{R}^n} (\delta^2 \bar{u}(x, y) - \delta^2 \bar{u}(0, y))^+ \frac{2-\sigma}{|y|^{n+\sigma}} dy \leq C \quad \text{for all } x \in B_1.$$

Translating this from \bar{u} to u we obtain that $0 \leq P \leq C\rho^\alpha$ in B_ρ and hence letting $\rho = R$ we obtain the bound for P in B_R of (2.4). The bounds for N in (2.6) are obtained likewise.

Next, dividing u by the universal constant C in (2.4) we may assume

$$0 \leq P \leq 2^{k\alpha} \leq 2^{k\bar{\alpha}} \quad \text{in } B_{2^k}(0) \quad \text{for all } k \geq 0. \quad (2.6)$$

In order to show that $u \in C^{\sigma+\bar{\alpha}}$ we will prove that

$$0 \leq P \leq 2^{k\bar{\alpha}} \quad \text{in } B_{2^k}(0) \quad \text{for all } k \leq -1. \quad (2.7)$$

This estimate on P is proved though an iterative improvement on the maximum of P on dyadic balls.

Indeed, our goal is to improve the bound from above $P \leq 1$ in B_1 to $P \leq 1 - \theta$ in $B_{1/2}$, for some $\theta > 0$. After doing this, we will immediately have (2.7) for all $k \geq 1$ for some $\bar{\alpha}$ small (related to θ) just by scaling and iterating. Let us thus concentrate in proving $P \leq 1 - \theta$ in $B_{1/2}$.

Let $x_0 \in B_{1/2}$ be such that $P(x_0) = \max_{B_{1/2}} P$. Define the set

$$A = \{y : (u(x_0 + y) + u(x_0 - y) - 2u(x_0) - u(y) - u(-y) + 2u(0)) > 0\}.$$

In particular we have

$$\begin{aligned} P(x_0) &= \int_A (\delta^2 u(x_0, y) - \delta^2 u(0, y)) \frac{2 - \sigma}{|y|^{n+\sigma}} dy, \\ N(x_0) &= \int_{\mathbb{R}^n \setminus A} (\delta^2 u(x_0, y) - \delta^2 u(0, y)) \frac{2 - \sigma}{|y|^{n+\sigma}} dy. \end{aligned}$$

We will take $\bar{\alpha}$ very small (depending on δ_0 below) so that (2.6) implies

$$\int_{\mathbb{R}^n} (P(y) - 1)^+ \frac{2 - \sigma}{|y|^{n+\sigma}} dy \leq \delta_0. \quad (2.8)$$

We define the function v as

$$v(x) := \int_A (\delta^2 u(x, y) - \delta^2 u(0, y)) \frac{2 - \sigma}{|y|^{n+\sigma}} dy.$$

Note that in particular $P(x_0) = v(x_0)$. Let

$$\bar{\theta} = \frac{\lambda}{4\Lambda} \quad (2.9)$$

and define the set

$$\mathbf{D} := \{x \in B_1 : v \geq (1 - \bar{\theta})\}.$$

Let us show that, for $\eta > 0$ small enough we have

$$|\mathbf{D}| \leq (1 - \eta)|B_1|. \quad (2.10)$$

Assume by contradiction that $|\mathbf{D}| \geq (1 - \eta)|B_1|$ for η small to be chosen later. That is, v is larger than $(1 - \bar{\theta})$ in most of B_1 . In that case we consider the function w defined as v but replacing A by $\mathbb{R}^n \setminus A$.

$$w(x) := \int_{\mathbb{R}^n \setminus A} (\delta^2 u(x, y) - \delta^2 u(0, y)) \frac{2 - \sigma}{|y|^{n+\sigma}} dy.$$

Using (iii), approximating $\chi_{\mathbb{R}^n \setminus A}(y)(2 - \sigma)|y|^{-n-\sigma}$ by L^1 functions μ with compact support and using the stability under uniform convergence result for subsolutions [4, Lemma 4.3] we show that

$$M_{\mathcal{L}_0}^+ w \geq 0 \quad \text{in } \mathbb{R}^n.$$

We observe that by definition $P - N = v + w$ and that, we have

$$0 \leq P - v \leq 1 - (1 - \bar{\theta}) \leq \bar{\theta} \quad \text{in } \mathbf{D}$$

—here we have used that $P \leq 1$ in B_1 by (2.6). Note in addition that the assumption (ii) yields

$$\frac{\lambda}{\Lambda} P(x) \leq N(x) \leq \frac{\Lambda}{\lambda} P(x). \quad (2.11)$$

Therefore,

$$\begin{aligned} w &= (P - v) - N \leq \bar{\theta} - N \leq \bar{\theta} - \frac{\lambda}{\Lambda} P \\ &\leq \bar{\theta} - \frac{\lambda}{\Lambda} (1 - \bar{\theta}) \\ &\leq -\lambda/\Lambda + 2\bar{\theta} \leq -c \quad \text{in } \mathbf{D}, \end{aligned}$$

where $c = \lambda/2\Lambda > 0$. Here we have used (2.9).

We now use the “half” Harnack of Theorem 5.1 in [5] applied to the function $\bar{w} = (w(r \cdot) + c)^+$ (with $r > 0$ small) to conclude that $w(0) + c \leq c/2$. Indeed, the function \bar{w} is a subsolution and, by (2.6), it satisfies $0 \leq P \leq 2^{k\bar{\alpha}}$ in $B_{2^k/r}(0)$ and $\bar{w} = 0$ in \mathbf{D}/r , which covers most of $B_{1/r}$. Hence, taking both r and η small enough we can make $\int_{\mathbb{R}^n} \bar{w}(y) \omega_\sigma(y) dy$ as small as we wish. Thus, using Theorem 5.1 in [5] we find that $w(0) + c = \bar{w}(0) \leq c/2$ as promised. As a consequence we obtain that $w(0) \leq -c/2 < 0$; a contradiction since $w(0) = 0$ by definition. This proves that (2.10) holds for some $\eta > 0$.

Note now that (2.10) is equivalent to

$$|\{x \in B_1 : v \leq (1 - \bar{\theta})\}| \geq \eta|B_1|.$$

Next, by (iii), approximating $\chi_A(y)|y|^{-n-\sigma}$ by L^1 functions μ with compact support and using the stability under uniform convergence result for subsolutions [4, Lemma 4.3] we show that

$$M_{\mathcal{L}_0}^+ v \geq 0 \quad \text{in } \mathbb{R}^n.$$

Taking now δ_0 small enough in (2.8) and using the L^ε Lemma of Theorem 10.4 in [3] applied to the function $(1-v)^+$, which nonnegative in all of \mathbb{R}^n and which is an approximate supersolution in $B_{3/4}$, we obtain that

$$1 - v \geq \bar{\theta}/C \quad \text{in all } B_{1/2}.$$

This is equivalent to saying

$$v \leq 1 - \bar{\theta}/C =: 1 - \theta \quad \text{in all } B_{1/2},$$

as we wanted to show. This proves (2.7).

We next note that (2.7) implies

$$0 \leq P(x) \leq C|x|^{\bar{\alpha}} \quad \text{for all } x \in B_1. \quad (2.12)$$

Given that (2.11) holds —recall that this follows from assumption (ii)— we similarly obtain that $0 \leq N(x) \leq C|x|^{\bar{\alpha}}$.

Finally we notice that the point 0 in the definition of P and N can be replaced by any point z in $B_{1/2}$. Therefore, using that

$$P(h) - N(h) = c(-\Delta)^{\sigma/2}(u(\cdot + h) - u)(0),$$

for some constant $c < 0$, we have shown —replacing 0 by any $z \in B_{1/2}$ — that

$$|(-\Delta)^{\sigma/2}(u(\cdot + h) - u)| \leq C|h|^{\bar{\alpha}} \quad \text{in } B_{1/2},$$

for all $h \in B_{1/4}$. This and the classical $C^{\bar{\alpha}}$ to $C^{\sigma+\bar{\alpha}}$ estimate for the Riesz potential $(-\Delta)^{-\sigma/2}$ easily imply that $[u]_{C^{\sigma+\bar{\alpha}}(B_{1/4})} \leq C$.

The same argument repeated at every scale —replacing u by the rescaled function $\bar{u} = \rho^{-\sigma-\alpha}u(\rho \cdot)$ for all $\rho \geq 1$ — yields $[\bar{u}]_{C^{\sigma+\bar{\alpha}}(B_{1/4})} \leq C$ which after rescaling gives (2.3). Then, as explained previously in this proof, the Theorem follows straightforward letting $R \rightarrow \infty$. \square

3. PRELIMINARY RESULTS AND PROOF OF THEOREM 1.1

The following proposition is the core of Theorem 1.1. It is in its proof (by contradiction) where we use the blow-up argument and the Liouville theorem described in the introduction.

Proposition 3.1. *Let $\sigma \in (0, 2)$. There is $\bar{\alpha} > 0$ (depending only on σ , ellipticity constants, and dimension) such that the following statement holds. Given $\alpha \in (0, \bar{\alpha})$ let ν be the floor of $\sigma + \alpha$ and assume that $\alpha' \in (0, \alpha)$ satisfies $\nu < \sigma + \alpha' < \sigma + \alpha$. Let $u \in C^{\sigma+\alpha'}(\mathbb{R}^n)$ be a solution of*

$$\inf_{a \in \mathcal{A}} (L_a u + c_a(x)) = 0 \quad \text{in } B_1,$$

where $\{L_a\} \subset \mathcal{L}_0(\sigma, \lambda, \Lambda)$. Assume that

$$\sup_{x \in B_1} \left| \inf_{a \in \mathcal{A}} c_a(x) \right| < +\infty \quad \text{and} \quad \sup_{a \in \mathcal{A}} [c_a]_{C^\alpha(B_1)} \leq C_0. \quad (3.1)$$

Then, $u \in C^{\sigma+\alpha}(\overline{B_{1/2}})$ and

$$[u]_{C^{\sigma+\alpha}(B_{1/2})} \leq C(\|u\|_{C^{\sigma+\alpha'}(\mathbb{R}^n)} + C_0),$$

where C_0 is the constant from (3.1) and C depends only on $n, \sigma, \alpha, \alpha', \lambda$, and Λ .

We will use the following trivial Claim.

Claim 3.2. *Let $\beta > 0$ and $\beta' \in (0, \beta)$. Let $\nu = \lfloor \beta \rfloor$ be the floor (or integer part) of β and assume that $\nu < \beta' < \beta$. Let u be a continuous function belonging to $C^{\beta'}(\mathbb{R}^n)$.*

If there exists $C_0 > 0$ such that

$$\sup_{r>0} \sup_{z \in B_{1/2}} r^{\beta'-\beta} [u]_{C^{\beta'}(B_r(z))} \leq C_0, \quad (3.2)$$

then

$$[u]_{C^\beta(B_{1/2})} \leq C_0. \quad (3.3)$$

Proof. It is enough to prove it for $\nu = 0$, that is, $0 < \beta' < \beta < 1$ since the result for $\nu \geq 1$ follows from this case applied to partial derivatives of u .

To prove it, note that (3.2) implies that for all $z \in B_{1/2}$ and for all $r > 0$ we have

$$\|u(z + \cdot) - u(z)\|_{L^\infty(B_r)} \leq r^{\beta'} [u]_{C^{\beta'}(B_r(z))} \leq r^{\beta'} C_0 r^{\beta-\beta'} = C_0 r^\beta.$$

Hence (3.3) follows. \square

We now give the

Proof of Proposition 3.1. The proof is by contradiction. If the statement of the proposition is false then, for each integer $k \geq 0$, there exist u_k and $C_{0,k}$ such that

- $\inf_{a \in \mathcal{A}_k} (L_a u + c_a(x)) = 0$ in B_1 ;
- $|\inf_{a \in \mathcal{A}_k} c_a(x)| < +\infty$ and $\sup_{a \in \mathcal{A}_k} [c_a]_{C^\alpha(B_1)} \leq C_{0,k}$;
- $\|u_k\|_{C^{\sigma+\alpha'}(\mathbb{R}^n)} + C_{0,k} \leq 1$ (we may always assume this dividing u_k by the previous quantity);

and

$$[u_k]_{C^{\sigma+\alpha}(B_{1/2})} \geq k.$$

Using Claim 3.2 with $\beta = \sigma + \alpha$ and $\beta' = \sigma + \alpha'$ we obtain that

$$\sup_k \sup_{z \in B_{1/2}} \sup_{r>0} r^{\alpha'-\alpha} [u_k]_{C^{\sigma+\alpha'}(B_r(z))} = +\infty. \quad (3.4)$$

Next we define

$$\theta(r) := \sup_k \sup_{z \in B_{1/2}} \sup_{r'>r} (r')^{\alpha'-\alpha} [u_k]_{C^{\sigma+\alpha'}(B_{r'}(z))},$$

The function θ is monotone nonincreasing and we have $\theta(r) < +\infty$ for $r > 0$ since we are assuming that $[u_k]_{C^{\sigma+\alpha'}(\mathbb{R}^n)} \leq 1$. In addition, by (3.4) we have $\theta(r) \nearrow +\infty$

as $r \searrow 0$. For every positive integer m , by definition of $\theta(1/m)$ there are $r'_m \geq 1/m$, k_m , and $z_m \in B_{1/2}$, for which

$$(r'_m)^{\alpha'-\alpha} [u_{k_m}]_{C^{\sigma+\alpha'}(B_{r'_m}(z_m))} \geq \frac{1}{2} \theta(1/m) \geq \frac{1}{2} \theta(r'_m). \quad (3.5)$$

Here we have used that θ is non-increasing. Note we will have $r'_m \searrow 0$.

Let $p_{k,z,r}(\cdot - z)$ be the polynomial of degree less or equal than ν in the variables $(x - z)$ which best fits u_k in $B_r(z)$ by least squares. That is,

$$p_{k,z,r} := \arg \min_{p \in \mathcal{P}_\nu} \int_{B_r(z)} (u_k(x) - p(x - z))^2 dx,$$

where \mathcal{P}_ν denotes the linear space of polynomials of degree at most ν with real coefficients. From now on in this proof we denote

$$p_m = p_{k_m, z_m, r'_m}.$$

We consider the blow up sequence

$$v_m(x) = \frac{u_{k_m}(z_m + r'_m x) - p_m(r'_m x)}{(r'_m)^{\sigma+\alpha} \theta(r'_m)}. \quad (3.6)$$

Note that, for all $m \geq 1$ we have

$$\int_{B_1(0)} v_m(x) q(x) dx = 0 \quad \text{for all } q \in \mathcal{P}_\nu. \quad (3.7)$$

This is the optimality condition for least squares.

Note also that (3.5) implies the following inequality for all $m \geq 1$:

$$\begin{aligned} [v_m]_{C^{\sigma+\alpha'}(B_1)} &= (r'_m)^{\sigma+\alpha'} \left[\frac{u_{k_m}(z_m + r'_m \cdot) - p_m(r'_m \cdot)}{(r'_m)^{\sigma+\alpha} \theta(r'_m)} \right]_{C^{\sigma+\alpha'}(B_{r'_m}(z_m))} \\ &= \frac{(r'_m)^{\alpha'-\alpha}}{\theta(r'_m)} [u_{k_m}(z_m + r'_m x)]_{C^{\sigma+\alpha'}(B_{r'_m}(z_m))} \geq 1/2, \end{aligned} \quad (3.8)$$

Here we have used that $\nu := \lfloor \sigma + \alpha \rfloor < \sigma + \alpha'$, and thus

$$[p_m(z_m + r'_m x)]_{C^{\sigma+\alpha'}(B_{r'_m}(z_m))} = 0,$$

since p_m is a polynomial of degree ν . Note that it is here where we crucially use the assumption that $\sigma + \alpha$ is not an integer.

Next we want to estimate

$$\begin{aligned} [v_m]_{C^{\sigma+\alpha'}(B_R)} &= \frac{1}{\theta(r'_m)(r'_m)^{\alpha-\alpha'}} [u_{k_m}]_{C^{\sigma+\alpha'}(B_{Rr'_m}(z_m))} \\ &= \frac{R^{\alpha-\alpha'}}{\theta(r'_m)(Rr'_m)^{\alpha-\alpha'}} [u_{k_m}]_{C^{\sigma+\alpha'}(B_{Rr'_m}(z_m))}. \end{aligned}$$

To do it, we use the definition of θ and its monotonicity to obtain the following growth control for the $C^{\sigma+\alpha'}$ seminorm of v_m

$$[v_m]_{C^{\sigma+\alpha'}(B_R)} \leq CR^{\alpha-\alpha'} \quad \text{for all } R \geq 1. \quad (3.9)$$

When $R = 1$, (3.9) implies that $\|v_m - q\|_{L^\infty(B_1)} \leq C$, for some $q \in \mathcal{P}_\nu$. Then, (3.7) implies that

$$\|v_m\|_{L^\infty(B_1)} \leq C. \quad (3.10)$$

Then, using (3.9) we obtain

$$[v_m]_{C^\beta(B_R)} \leq CR^{\sigma+\alpha-\beta}. \quad (3.11)$$

for all $\beta \in [0, \sigma + \alpha']$. Indeed, (3.10) implies that for every multiindex l with $|l| \leq \nu$ there is some point $x_* \in B_1$ such that $|D^l v_m(x_*)| \leq C$. The existence of such x_* can be shown taking some nonnegative $\eta \in C_c^\infty(B_1)$ with unit mass and observing that the inequality

$$\left| \int \eta(x) D^l v_m(x) dx \right| \leq C \int |D^l \eta| v_m(x) dx \leq C$$

rules out the two possibilities $D^l v_m > C$ and $D^l v_m < -C$ in all of B_1 .

Hence, using (3.9), we obtain that for all l with $|l| = \nu$ and $x \in B_R$ we have

$$|D^l v_m(x)| \leq |D^l v_m(x^*)| + CR^{\alpha-\alpha'} |x - x^*|^{\sigma+\alpha'-\nu} \leq CR^{\sigma+\alpha-\nu}.$$

Iterating the same argument, we then show the corresponding estimate for all l with $0 \leq |l| \leq \nu$. Then (3.11) for all $\beta \in [0, \sigma + \alpha']$ follows by interpolation.

We now claim that, by further rescaling v_m if necessary, we may assume that in addition to (3.8) the following holds

$$\sup_{|l|=\nu} \text{osc}_{B_1} D^l v_m \geq 1/4, \quad (3.12)$$

where l donates a multiindex. Indeed, if (3.8) holds then there are $x_m \in B_1$ and $h_m \in B_{1-|x_m|}$ such that

$$\sup_{|l|=\nu} \frac{|D^l v_m(x_m + h_m) - D^l v_m(x_m)|}{|h_m|^{\sigma+\alpha'-\nu}} \geq 1/4$$

and thus we can consider, instead of v_m , the function

$$\tilde{v}_m = \frac{v_m(x_m + |h_m|x) - \tilde{p}_m(x)}{|h_m|^{\sigma+\alpha'}},$$

where $\tilde{p}_m \in \mathcal{P}_\nu$ is chosen so that \tilde{v}_m satisfies (3.7) (with v_m replaced by \tilde{v}_m).

Note that \tilde{p}_m is the polynomial that approximates better (in the L^2 sense) $v_m(x_m + \cdot)$ in $B_{|h_m|}(x_m)$ and since $v_m \in C^{\sigma+\alpha'}$ with the control (3.9) we have

$$|v_m(x_m + |h_m|x) - \tilde{p}_m(x)| \leq C|h_m|^{\sigma+\alpha'}|x|^{\sigma+\alpha'}.$$

Therefore, \tilde{v}_m also satisfies (3.9) and (3.11) (with v_m replaced by \tilde{v}_m). Note that \tilde{v}_m would also be of the form (3.6) for new z_m and r'_m defined as $z_m + x_m$ and $|h_m|r'_m$, respectively —where we use that $\theta(|h_m|r'_m) \geq \theta(r'_m)$.

In summary, the new sequence \tilde{v}_m satisfies the same properties as v_m and, in addition, (3.12), as desired.

Next we prove the following

Claim. A subsequence of v_m converges in $C^{(\nu+\sigma+\alpha')/2}_{\text{loc}}(\mathbb{R}^n)$ to a function $v \in C^{\sigma+\alpha'}_{\text{loc}}(\mathbb{R}^n)$. This function v satisfies the assumptions of the Liouville-type Theorem 2.1.

The $C^{(\nu+\sigma+\alpha')/2}$ uniform convergence on compact sets of \mathbb{R}^n of a subsequence of v_m to some $v \in C^{\sigma+\alpha'}(\mathbb{R}^n)$ follows from (3.11) and the Arzelà-Ascoli theorem (and the typical diagonal sequence trick) —note that since $\nu < \sigma + \alpha'$ the exponent $(\nu + \sigma + \alpha')/2$ is less than $\sigma + \alpha'$, as required to have compactness in the norm $C^{(\nu+\sigma+\alpha')/2}$ of a equibounded sequence in the stronger norm $C^{\sigma+\alpha'}$. The only important fact about the election of the exponent $(\nu + \sigma + \alpha')/2$ is that it is greater than ν and σ .

First, passing to the limit (3.11) we find that the assumption (i) of Theorem 2.1 is satisfied by this limit function v .

Now, each u_k satisfies a concave equation of the type (1.7)-(1.8)-(1.10). Thus, for every L^1 density $d\mu(h)$ with compact support and $\mu(\mathbb{R}^n) = 1$ and for m large enough we have

$$\begin{aligned} 0 &= \int d\mu\left(\frac{\bar{h}}{r'_m}\right) \inf_{a \in \mathcal{A}_{k_m}} (L_a u_{k_m}(\bar{x} + \bar{h}) + c_a(\bar{x} + \bar{h})) - 0 \\ &\leq \inf_{a \in \mathcal{A}_{k_m}} \left(\int L_a u_{k_m}(\bar{x} + \bar{h}) + c_a(\bar{x} + \bar{h}) d\mu\left(\frac{\bar{h}}{r'_m}\right) \right) - \inf_{a \in \mathcal{A}_{k_m}} (L_a u_{k_m}(\bar{x}) + c_a(\bar{x})). \end{aligned}$$

Recall that by (3.1) we have $\sup_{a \in \mathcal{A}_k} [c_a]_{C^\alpha(B_1)} \leq C_{0,k}$. Hence, for all $\bar{x} \in B_{3/4}(z)$ provided that m is chosen large enough so that $r'_m \text{diam}(\text{supp } \mu) \leq 1/4$ we have

$$\begin{aligned} -C_{0,k_m} \int d\mu\left(\frac{\bar{h}}{r'_m}\right) |\bar{h}|^\alpha &\leq - \sup_{a \in \mathcal{A}_{k_m}} [c_a]_{C^\alpha(B_1)} \int d\mu\left(\frac{\bar{h}}{r'_m}\right) |\bar{h}|^\alpha \\ &\leq \inf_{a \in \mathcal{A}_{k_m}} \left(\int L_a u_{k_m}(\bar{x} + \bar{h}) d\mu\left(\frac{\bar{h}}{r'_m}\right) + c_a(\bar{x}) \right) \\ &\quad - \inf_{a \in \mathcal{A}_{k_m}} (L_a u_{k_m}(\bar{x}) + c_a(\bar{x})) \\ &\leq \sup_{a \in \mathcal{A}_{k_m}} \left(\int L_a u_{k_m}(\bar{x} + \bar{h}) d\mu\left(\frac{\bar{h}}{r'_m}\right) - L_a u_{k_m}(\bar{x}) \right) \\ &\leq M_{\mathcal{L}_0}^+ \left(\int u_{k_m}(\cdot + \bar{h}) d\mu\left(\frac{\bar{h}}{r'_m}\right) - u_{k_m} \right)(\bar{x}). \end{aligned} \tag{3.13}$$

Note now that, since $\nu \leq 2$,

$$\delta^2 p(x + h, y) - \delta^2 p(x, y) = 0 \quad \text{for all } p \in \mathcal{P}_\nu \quad \text{and for all } x, y, h \text{ in } \mathbb{R}^n. \tag{3.14}$$

Taking into account (3.14), we now translate (3.13) from u_{k_m} to v_m . Using the definition of v_m in (3.6), and setting $\bar{h} = r'_m h$ and $\bar{x} = z_m + r'_m x$ in (3.13), we obtain

$$\begin{aligned} -C_{0,k_m}(r'_m)^\alpha \int_{\mathbb{R}^n} d\mu(h) |h|^\alpha &\leq \\ &\leq \frac{1}{(r'_m)^\sigma} M_{\mathcal{L}_0}^+ \left((r'_m)^{\sigma+\alpha} \theta(r'_m) \left\{ \int v_m(\cdot + h) d\mu(h) - v_m \right\} \right) (x) \\ &\leq (r'_m)^\alpha \theta(r'_m) M_{\mathcal{L}_0}^+ \left(\int v_m(\cdot + h) d\mu(h) - v_m \right) (x) \end{aligned}$$

whenever $|x| \leq \frac{1}{Cr'_m}$, and thus

$$-\frac{C_{0,k_m}}{\theta(r'_m)} C(\mu) \leq M_{\mathcal{L}_0}^+ \left(\int v_m(\cdot + h) d\mu(h) - v_m \right) \quad \text{in } |x| \leq \frac{1}{Cr'_m}. \quad (3.15)$$

Given that μ has compact support, that $|C_{0,k_m}| \leq 1$, and that $\theta(r'_m) \nearrow \infty$, we obtain that the left hand side of (3.15) converges to zero $m \rightarrow +\infty$. Thus, passing (3.15) to the limit we find that

$$0 \leq M_{\mathcal{L}_0}^+ \left(\int v(\cdot + h) d\mu(h) - v \right) \quad \text{in all of } \mathbb{R}^n.$$

Indeed, to carefully justify the previous limit $m \rightarrow +\infty$ on the right hand side of (3.15) we are using that, by (3.11), the functions

$$w_m := \int v_{k_m}(\cdot + h) d\mu(h) - v_{k_m}$$

satisfy, for all $R \geq \text{diam}(\text{supp } \mu)$, that

$$[w_m]_{C^{\sigma+\alpha'}(B_R)} \leq CR^{\alpha-\alpha'} \quad \text{and} \quad |w_m(x)| \leq C \int |h|^\beta d\mu(h) |x|^{\sigma+\alpha-\beta} \leq C|x|^{\sigma+\alpha-\beta}.$$

Thus, taking $\beta \in (\alpha, \min\{1, \sigma + \alpha'\})$, and since $|x|^{\sigma+\alpha-\beta} \in L^1(\mathbb{R}^n, \omega_\sigma)$, we can use the dominated convergence theorem to compute the limit. Therefore, the assumption (iii) of Theorem 2.1 is satisfied by v .

A very similar (actually easier) computation shows that the assumption (ii) is also satisfied by v . This finishes the proof the Claim.

We have thus proved that v satisfies all the assumptions of Theorem 2.1 and hence we conclude that v is a polynomial of degree ν . On the other hand, passing (3.7) to the limit we obtain that v is orthogonal to every polynomial of degree ν in B_1 , and hence it must be $v \equiv 0$. But then passing (3.12) to the limit we obtain that v cannot be constantly zero in B_1 ; a contradiction. \square

Using Proposition 3.1 we prove an intermediate technical statement that will be later used to prove Theorem 1.1.

Proposition 3.3. *Let $\sigma \in (0, 2)$, λ , Λ , A_0 and C_0 be given constants with $0 < \lambda \leq \Lambda$. Suppose that $A_0 \leq 1$. There is $\bar{\alpha} > 0$ (depending only on n , σ , λ and Λ) such that the following statement holds. Given α and α' satisfying $0 < \alpha' < \alpha < \bar{\alpha}$ and $\nu < \sigma + \alpha' < \sigma + \alpha$, where ν is the floor of $\sigma + \alpha$. Let $u \in C^{\sigma+\alpha}(\overline{B_1}) \cap C^\alpha(\mathbb{R}^n)$ be a solution of*

$$I(u, x) = 0 \quad \text{in } B_1,$$

where I , defined by (1.7), satisfies (1.8), (1.9), and (1.10).

Then,

$$[u]_{C^{\sigma+\alpha}(B_{1/4})} \leq C(\|u\|_{C^{\sigma+\alpha'}(B_1)} + A_0\|u\|_{C^{\sigma+\alpha}(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)} + C_0), \quad (3.16)$$

where C_0 is the constant from (1.10) and C depends only on n , σ , α , α' , λ , Λ .

Proof. Let $\eta \in C_c^\infty(B_1)$ be a cutoff function satisfying $\eta \equiv 1$ in $B_{3/4}$. Then,

$$\|\eta u\|_{C^{\sigma+\alpha'}(\mathbb{R}^n)} \leq C\|u\|_{C^{\sigma+\alpha'}(B_1)}. \quad (3.17)$$

In addition, we have

$$0 = I(\eta u + (1 - \eta)u, x) = \inf_{a \in \mathcal{A}} \left(\int_{\mathbb{R}^n} \delta^2(\eta u)(x, y) K_a(0, y) dy + \tilde{c}_a(x) \right) \quad (3.18)$$

where

$$\tilde{c}_a(x) = c_a(x) + \mathbf{A}(x) + \mathbf{B}(x),$$

for

$$\mathbf{A}(x) = \int_{\mathbb{R}^n} \delta^2(\eta u)(x, y) (K_a(x, y) - K_a(0, y)) dy,$$

and

$$\mathbf{B}(x) = \int_{\mathbb{R}^n} \delta^2((1 - \eta)u)(x, y) K_a(x, y) dy$$

We next write for $x, x' \in B_{1/2}$,

$$\begin{aligned} \mathbf{A}(x) - \mathbf{A}(x') &= \int_{\mathbb{R}^n} \delta^2(\eta u)(x', y) (K_a(x, y) - K_a(x', y)) dy + \\ &\quad + \int_{\mathbb{R}^n} (\delta^2(\eta u)(x, y) - \delta^2(\eta u)(x', y)) (K_a(x, y) - K_a(0, y)) dy. \\ &= \mathbf{A}_1(x, x') + \mathbf{A}_2(x, x') \end{aligned}$$

Let us now bound $|\mathbf{A}(x) - \mathbf{A}(x')|$. We will do the case $\nu = \lfloor \sigma + \alpha \rfloor = 2$ (the cases $\nu = 1$ and $\nu = 0$ are very similar). On the one hand, we obtain

$$\begin{aligned} |\mathbf{A}_1(x, x')| &\leq \int_{\mathbb{R}^n} |\delta^2(\eta u)(x', y)| |K_a(x, y) - K_a(x', y)| dy \\ &\leq \int_{B_{1/2}} |y|^2 \|u\|_{C^{\sigma+\alpha}(B_1)} |K_a(x, y) - K_a(x', y)| dy + \\ &\quad + \int_{\mathbb{R}^n \setminus B_{1/2}} \|u\|_{L^\infty(\mathbb{R}^n)} |K_a(x, y) - K_a(x', y)| dy \\ &\leq CA_0 |x - x'|^\alpha (\|u\|_{C^{\sigma+\alpha}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}), \end{aligned}$$

where we have used (1.9).

On the other hand, letting $d = |x - x'|$ we have

$$|\delta^2(\eta u)(x, y) - \delta^2(\eta u)(x', y)| \leq \begin{cases} |y|^2 d^{\sigma+\alpha-2} \|u\|_{C^{\sigma+\alpha}(B_1)} & y \text{ in } B_d \\ d^2 |y|^{\sigma+\alpha-2} \|u\|_{C^{\sigma+\alpha}(B_1)} & y \text{ in } B_{1/2} \setminus B_d \\ d^\alpha \|u\|_{C^\alpha(\mathbb{R}^n)} & y \text{ outside } B_{1/2}. \end{cases}$$

Combining this and (1.9) we readily obtain

$$\begin{aligned} |\mathbf{A}_2(x, x')| &\leq \int_{\mathbb{R}^n} |\delta^2(\eta u)(x, y) - \delta^2(\eta u)(x', y)| |K_a(x, y) - K_a(0, y)| dy \\ &\leq Cd^\alpha A_0 |x - 0|^\alpha (\|u\|_{C^{\sigma+\alpha}(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}), \end{aligned}$$

Hence,

$$[\mathbf{A}]_{C^\alpha(B_{1/2})} \leq CA_0 (\|u\|_{C^{\sigma+\alpha}(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}). \quad (3.19)$$

On the other hand, letting $\tilde{u} = (1 - \eta)u$ and using that $\tilde{u} \equiv 0$ in $B_{3/4}$, we obtain with similar computations

$$\begin{aligned} |\mathbf{B}(x) - \mathbf{B}(x')| &\leq \int_{\mathbb{R}^n} |\delta^2 \tilde{u}(x, y) - \delta^2 \tilde{u}(x', y)| |K(x, y)| dy + \\ &\quad + \int_{\mathbb{R}^n} |\delta^2 \tilde{u}(x, y)| |K_a(x, y) - K_a(x', y)| dy \\ &\leq C(\Lambda + A_0) |x - x'|^\alpha \|u\|_{C^\alpha(\mathbb{R}^n)}. \end{aligned}$$

Hence,

$$[\mathbf{B}]_{C^\alpha(B_{1/2})} \leq C(\Lambda + A_0) \|u\|_{C^\alpha(\mathbb{R}^n)}. \quad (3.20)$$

Therefore, using (3.19) and (3.20), and recalling that we assume that $A_0 \leq 1$, we obtain

$$[\tilde{c}_a(x)]_{C^\alpha(B_{1/2})} \leq C_0 + CA_0 \|u\|_{C^{\sigma+\alpha}(B_1)} + C \|u\|_{C^\alpha(\mathbb{R}^n)} \quad (3.21)$$

where C depends only on n, σ, λ and Λ .

We have thus proven that the function ηu belongs to $C^{\sigma+\alpha'}(\mathbb{R}^n)$ with the control (3.17) on this norm and solves the equation (3.24) in $B_{1/2}$ with $\tilde{c}(x)$ satisfying (3.21).

Hence, ηu satisfies the assumptions of Proposition 3.1 and therefore (3.16) follows from the estimate provided by the same proposition. \square

As a last ingredient for the proof of Theorem 1.1, we recall the adimensional Hölder seminorms from the classical book Gilbarg-Trudinger [7]. We next recall the definition of the adimensional C^β seminorm from Section 4 of [7]. Let $\beta > 0$ and let k be the integer such that $\beta = k + \beta'$ for some $\beta' \in (0, 1]$. Then,

$$[u]_{\beta;\Omega}^* = \sup_{x,y \in \Omega, |l|=k} (d_{x,y})^\beta \frac{|D^l u(x) - D^l u(y)|}{|x - y|^{\beta'}},$$

where $d_{x,y} := \min\{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\}$.

We next give the

Proof of Theorem 1.1. Let $\rho \in (0, 1)$ and $z \in B_1$ be such that $B_\rho(z) \subset B_1$. Let $\bar{u}(\bar{x}) = u(z + \rho\bar{x})$. The function \bar{u} solves in B_1 the rescaled equation

$$\bar{\mathbf{I}}(\bar{u}, \bar{x}) = \min_a \left(\int_{\mathbb{R}^n} \delta^2 \bar{u}(\bar{x}, \bar{y}) \rho^{n+\sigma} K_a(z + \rho\bar{x}, \rho\bar{y}) d\bar{y} + \rho^\sigma c_a(z + \rho\bar{x}) \right) = 0 \quad (3.22)$$

in B_1 . Note that if the kernels $K_a(x, y)$ of the original operator \mathbf{I} satisfy (1.8)-(1.9)-(1.10), then the rescaled kernels

$$\bar{K}_a(\bar{x}, \bar{y}) := \rho^{n+\sigma} K_a(z + \rho\bar{x}, \rho\bar{y})$$

of $\bar{\mathbf{I}}$ also satisfies (1.8)-(1.9)-(1.10) with the same constants $\lambda, \Lambda, A_0, C_0$ as those of \mathbf{I} . In fact, we have

$$\begin{aligned} \int_{B_{2r} \setminus B_r} |\bar{K}_a(\bar{x}, \bar{y}) - \bar{K}_a(\bar{x}', \bar{y})| d\bar{y} &= \\ &= \int_{B_{2\rho r} \setminus B_{\rho r}} \rho^{n+\sigma} |K_a(z + \rho\bar{x}, y) - K_a(z + \rho\bar{x}', y)| \frac{dy}{\rho^n} \\ &\leq A_0 |\rho(\bar{x} - \bar{x}')|^\alpha \frac{2 - \sigma}{r^\sigma} \end{aligned}$$

Hence, as it will be used on in this proof, $\bar{\mathbf{I}}$ satisfies (1.9) with A_0 replaced by $\rho^\alpha A_0 \leq A_0$.

Let $\nu = \lfloor \sigma + \alpha \rfloor$ and $\alpha' = \alpha'(\sigma, \alpha, \nu)$ be given by (2.1). Since $\sigma + \alpha > \nu$ by assumption (it is not an integer) we have $\alpha' \in (0, \alpha)$ and $\nu < \sigma + \alpha'$. Then, assuming that $A_0 \leq 1$, Proposition 3.3 applied to \bar{u} yields

$$[\bar{u}]_{C^{\sigma+\alpha}(B_{1/4})} \leq C \left(\|\bar{u}\|_{C^{\sigma+\alpha'}(B_1)} + A_0 \|\bar{u}\|_{C^{\sigma+\alpha}(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)} + C_0 \right), \quad (3.23)$$

where C_0 is the constant from (1.10) and C depends only on $n, \sigma, \alpha, \lambda, \Lambda$.

Using standard interpolation inequalities in B_1 to control the full norm $\|\cdot\|_{C^{\sigma+\alpha}(B_1)}$ by $[\cdot]_{C^{\sigma+\alpha}(B_1)} + \|\cdot\|_{L^\infty(B_1)}$, and scaling back (3.23) from \bar{u} to u , we obtain

$$\rho^{\sigma+\alpha} [u]_{C^{\sigma+\alpha}(B_{\rho/4}(z))} \leq C \left(\rho^{\sigma+\alpha'} [u]_{C^{\sigma+\alpha'}(B_\rho(z))} + A_0 \rho^{\sigma+\alpha} [u]_{C^{\sigma+\alpha}(B_\rho(z))} + \|u\|_{C^\alpha(\mathbb{R}^n)} + C_0 \right).$$

The previous estimate holds in every ball $B_\rho(z) \subset B_1$ and this immediately yields, in terms of the adimensional Hölder norms, that

$$[u]_{\sigma+\alpha;B_1}^* \leq C([u]_{\sigma+\alpha';B_1}^* + A_0[u]_{\sigma+\alpha;B_1}^* + \|u\|_{C^\alpha(\mathbb{R}^n)} + C_0).$$

Then, assuming that $A_0 \leq \varepsilon_0$, with ε_0 small enough (depending only on n, σ, λ , and Λ), and using the interpolation inequality for adimensional Hölder norms [7, Lemma 6.32 in Section 6.8]

$$[u]_{\sigma+\alpha';B_1}^* \leq \delta[u]_{\sigma+\alpha;B_1}^* + C(\delta)\|u\|_{L^\infty(B_1)}$$

we obtain

$$[u]_{\sigma+\alpha;B_1}^* \leq C(\|u\|_{C^\alpha(\mathbb{R}^n)} + C_0).$$

Since clearly $[u]_{C^{\sigma+\alpha}(B_{1/2})} \leq C[u]_{C^{\sigma+\alpha}(B_1)}^*$ we have proven the theorem in the case of $A_0 \leq \varepsilon_0 \ll 1$.

Let us prove now the Theorem also for non-small A_0 . We only need to use a typical scaling trick. Let as before $z \in B_{1/2}$ and $\rho \in (0, 1]$ such that $B_\rho(z) \subset B_1$. We have already seen that if the function $\bar{u} = u(z + \rho\bar{x})$ solves the rescaled equation (3.22) in B_1 and that A_0 in the new equation by $\rho^\alpha A_0$. Therefore, choosing ρ small enough—depending on A_0 and α —so that $\rho^\alpha A_0 \leq \varepsilon_0$ we may apply the previous estimate to the rescaled equation to obtain

$$[\bar{u}]_{C^{\sigma+\alpha}(B_{1/2})} \leq C(\|\bar{u}\|_{C^\alpha(\mathbb{R}^n)} + C_0)$$

and thus

$$[u]_{C^{\sigma+\alpha}(B_{\rho/2}(z))} \leq C\rho^{-\sigma-\alpha}(\|\bar{u}\|_{C^\alpha(\mathbb{R}^n)} + C_0).$$

Since a finite number of these balls cover $\overline{B_{1/2}}$, the estimate of the Theorem follows. \square

To end the section we give the

Proof of Corollary 1.2. First, note that using for instance the Hölder estimate in [3], the solution u belongs to $C^\alpha(\overline{B_{7/8}})$ with an estimate—note that $\alpha < \bar{\alpha}$ with $\bar{\alpha}$ small enough. Let $\eta \in C_c^\infty(B_1)$ be a smooth cutoff function with $\eta \equiv 1$ in $B_{6/8}$ and $\eta \equiv 0$ outside $B_{7/8}$.

Then, $\eta u \in C^\alpha(\mathbb{R}^n)$ solves the following equation in $B_{5/8}$:

$$0 = \mathbf{I}(\eta u + (1 - \eta)u, x) = \inf_{a \in \mathcal{A}} \left(\int_{\mathbb{R}^n} \delta^2(\eta u)(x, y) K_a(x, y) dy + \tilde{c}_a(x) \right) \quad (3.24)$$

where

$$\tilde{c}_a(x) = c_a(x) + \int_{\mathbb{R}^n} \delta^2((1 - \eta)u)(x, y) K_a(x, y) dy.$$

Using the additional assumption (1.11) and the fact that $(1 - \eta) \equiv 0$ in $B_{6/8}$ we readily show that $\|\tilde{c}_a\|_{C^\alpha(B_{5/8})} \leq C_0 + C\|u\|_{L^\infty(\mathbb{R}^n)}$. Therefore, under the assumptions of the Corollary, the function $\eta u \in C^\alpha(\mathbb{R}^n)$ solves an equation in $B_{5/8}$ that satisfies the assumptions of Theorem 1.1 with B_1 replaced by $B_{5/8}$. Then, applying the estimate of Theorem 1.1 (rescaled) to the function ηu the Corollary follows—since

the equation is satisfied in $B_{5/8}$ instead of B_1 we need to use the standard covering argument to obtain the estimate in $B_{1/2}$ instead of $B_{5/16}$. \square

4. APPROXIMATION PROCEDURE FOR NON TRANSLATION INVARIANT EQUATIONS

In this section we show a way of approximating a non translation invariant equation $I(u, x) = 0$ in B_1 of the form (1.7) and satisfying (1.8)-(1.9)-(1.10) by a sequence of equations $I^\epsilon(u^\epsilon, x) = 0$ that admit C^3 solutions in B_1 . This approximation procedure is modification of the one in [5].

For $\epsilon \in (0, 1)$, let

$$I^\epsilon(u^\epsilon, x) := \inf_{a \in \mathcal{A}} \left(\int_{\mathbb{R}} \delta^2 u^\epsilon(x, y) K_a^\epsilon(x, y) dy + c_a^\epsilon(x) \right) \quad (4.1)$$

where, for all $a \in \mathcal{A}$ and for all x in B_1 and $y \in \mathbb{R}^n \setminus \{0\}$, we have

$$\begin{aligned} K_a^\epsilon(x, y) = & \xi \left(\frac{y}{4\epsilon} \right) \frac{(2 - \sigma)}{|y|^{n+\sigma}} + \\ & + \left(1 - \xi \left(\frac{y}{4\epsilon} \right) \right) \int_{\mathbb{R}^n} \frac{d\bar{x}}{\epsilon^n} \int_{\mathbb{R}^n} \frac{d\bar{y}}{\epsilon^n} K_a(x - \bar{x}, y - \bar{y}) \eta \left(\frac{\bar{x}}{\epsilon} \right) \eta \left(\frac{\bar{y}}{\epsilon} \right), \end{aligned} \quad (4.2)$$

and

$$c_a^\epsilon(x) = \int_{\mathbb{R}^n} \frac{d\bar{x}}{\epsilon^n} c_a(x - \bar{x}) \eta \left(\frac{\bar{x}}{\epsilon} \right), \quad (4.3)$$

for some $\xi \in C_c^\infty(B_1)$ with $\xi \equiv 1$ in $B_{1/2}$ and for some $\eta \in C_c^\infty(B_1)$ with $\eta \geq 0$ and $\int \eta = 1$.

Remark 4.1. Note that the operator I^ϵ satisfies (1.8)-(1.9)-(1.10) —as I — with the same constants A_0 , C_0 , and with λ , Λ replaced by λ/C , $C\Lambda$ respectively. If in addition the operator I satisfies (1.11) then so does I^ϵ again with Λ being replaced by $C\Lambda$. Note in addition that $I^\epsilon \rightarrow I$ in weakly in B_1 and with the weight ω_σ —for the notion of weak convergence of nonlocal elliptic operators see [4].

We will prove the following

Proposition 4.2. *For all $\epsilon > 0$, the Dirichlet problem*

$$\begin{cases} I^\epsilon(u^\epsilon, x) = 0 & \text{in } B_1 \\ u = g & \text{outside } B_1 \end{cases} \quad (4.4)$$

with bounded $g \in C(\mathbb{R}^n \setminus B_1)$ admits a unique solution $u^\epsilon \in C(\mathbb{R}^n) \cap C^3(B_1)$.

Proof. To show that, for all $\epsilon > 0$ the Dirichlet problem (4.5) admits a unique solution $u \in C(\mathbb{R}^n) \cap C^3(B_1)$ we will use Perron's method. Since a comparison principle between viscosity solutions is not available for non translation invariant nonlocal fully nonlinear equations, the use of Perron's method will be based in the following crucial observation (existence of smooth solutions in tiny balls for the regularized equation).

Claim. *Given $\epsilon > 0$, there is $\delta_0 > 0$ with $\delta_0 \ll \epsilon$ such that whenever $B_\delta(z)$ is a ball contained in B_1 with $\delta \in (0, \delta_0)$ there exists a unique solution $w \in C(\mathbb{R}^n) \cap C^3(B_\delta)$ to the Dirichlet problem*

$$\begin{cases} I^\epsilon(w, x) = 0 & \text{in } B_\delta(z) \\ w = h & \text{in } \mathbb{R}^n \setminus B_\delta(z) \end{cases} \quad (4.5)$$

for all continuous complement data h with $\|h\|_{L^\infty(\mathbb{R}^n \setminus B_\delta)} \leq 1$.

Moreover, the function w satisfies

$$\|w\|_{C^3(B_\delta(z))} \leq C\|h\|_{L^\infty(\mathbb{R}^n)} \quad (4.6)$$

where C depends on $n, \lambda, \Lambda, \epsilon$, and δ .

To prove the Claim, for fixed $\epsilon > 0$ we rescale the operator I^ϵ as follows

$$I^\epsilon(w, z + \delta\bar{x}) = \delta^{-\sigma} \bar{I}(w(z + \delta \cdot), \bar{x}).$$

Note that the kernels that define the new operator \bar{I} are smooth and coincide to that of the fractional Laplacian inside of a large ball $B_{\epsilon/\delta}$ (recall that $\delta \ll \epsilon$). Hence, writing $C\bar{I}(\bar{w}, \bar{x}) = (-\Delta)^{\sigma/2} \bar{w}(x) + \mathcal{N}_\delta(\bar{w}, \bar{x})$ for the rescaled function $\bar{w} = w(z + \delta \cdot)$ the problem (4.5) takes the form

$$\begin{cases} -(-\Delta)^{\sigma/2} \bar{w} + \mathcal{N}_\delta(\bar{w}, \bar{x}) = 0 & \text{in } B_1 \\ \bar{w} = \bar{h} & \text{in } \mathbb{R}^n \setminus B_1, \end{cases} \quad (4.7)$$

where $x = z + \delta\bar{x}$,

$$\mathcal{N}_\delta(\bar{w}, \bar{x}) = C \inf_{a \in \mathcal{A}} \left(\int_{\mathbb{R}} \delta^2 u^\epsilon(x, \bar{y}) (\delta^{n+\sigma} K_a^\epsilon(\bar{x}, \delta y) - (2 - \sigma)|y|^{-n-\sigma}) dy + \delta^\sigma c_a^\epsilon(x) \right)$$

and $\bar{h}(\bar{x}) = h(z + \delta\bar{x})$.

Notice that

$$\delta^{n+\sigma} K_a^\epsilon(\bar{x}, \delta y) - (2 - \sigma)|y|^{-n-\sigma} \equiv 0 \quad \text{in } B_{2\epsilon/\delta}$$

by the definition of K_a^ϵ in (4.2). Then, it is straightforward to verify —using (4.2) and (4.3)—that

$$\|\mathcal{N}_\delta(w, \cdot)\|_{C^3(\overline{B_1})} \leq \gamma_\delta \|w\|_{L^\infty(\mathbb{R}^n)} \quad (4.8)$$

and

$$\|\mathcal{N}_\delta(w, x) - \mathcal{N}_\delta(w', x)\|_{L^\infty(B_1)} \leq \gamma_\delta \|w - w'\|_{L^\infty(\mathbb{R}^n)} \quad (4.9)$$

for every $w, w' \in L^\infty(\mathbb{R}^n)$ where

$$\gamma_\delta \searrow 0 \quad \text{as } (\delta/\epsilon) \searrow 0.$$

Therefore, a solution $w \in L^\infty(\mathbb{R}^n) \cap C^3(B_1)$ to (4.7) can be then constructed using the solvability of the Dirichlet problem with the fractional Laplacian and the Banach fixed point theorem. Indeed, let

$$\mathcal{I}_h[f] =: v$$

be the unique solution to

$$\begin{cases} (-\Delta)^{\sigma/2} v = f & \text{in } B_1 \\ v = \bar{h} & \text{in } \mathbb{R}^n \setminus B_1, \end{cases} \quad (4.10)$$

Then, (4.7) can be restated as a fixed point problem as

$$\mathcal{I}_{\bar{h}}[\mathcal{N}_{\delta}(w, \cdot)] = w.$$

The contractivity of the previous map in the “closed ball” $\{w : \|w\|_{L^\infty(B_1)} \leq 2\}$ when $\delta/\epsilon \ll 1$ follows from (4.8)-(4.9) and the elementary estimate for (4.10)

$$\|v\|_{L^\infty(B_1)} \leq \|\bar{h}\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + C\|f\|_{L^\infty(B_1)}.$$

The continuity up to the boundary of w —with implies the uniqueness of solution to (4.5) in the class of viscosity solutions— follows from the results in the Section 3 of [4]. This finishes the proof of the Claim.

The previous Claim makes now it simple to apply of Perron’s method to show existence of solution. As usual, we consider the following candidate to viscosity solution to (4.5):

$$u^\epsilon(x) = \sup\{w(x) : w \in C(\overline{\Omega}) \text{ is a viscosity subsolution of (4.5)}\}. \quad (4.11)$$

Using the Claim, and the barriers from Section 3 in [4], the ideas of the classical proof by Perron’s method of the existence of a harmonic function with given continuous boundary data in smooth domains apply to this case, since we also have solvability in balls (in our case tiny ones). We obtain that u^ϵ solves classically the equation in the interior and attains continuously the complement data.

Indeed, as for harmonic functions, in the supremum of (4.11) defining $u(x)$, for every $\delta \in (0, \delta_0)$ such that $B_\delta(z) \subset B_1$ we can replace the subsolution w by the solution in $B_\delta(z)$ with its same values outside. The new function will be larger by the comparison principle between a viscosity an a smooth solutions. It then follows using (4.6) and Arzelà-Ascoli that u^ϵ belongs $C^3(B_1)$, and that it is a solution to the equation in the interior of B_1 . That u^ϵ defined as in (4.11) is continuous function up to the boundary attaining the complement data follows from standard barrier arguments, employing the barriers from Section 3 of [3]. \square

The remaining part of this section will be devoted to the proof of Theorem (1.3). In it, we will need the following Proposition.

Proposition 4.3. *Let $\sigma \in (0, 2)$, and λ, Λ be given constants with $0 < \lambda \leq \Lambda$. Then, there exists $\gamma \in (0, 1)$ depending only on $n, \sigma, \lambda, \Lambda$ such that the following statement holds.*

Let $\alpha \in (0, \gamma)$ and assume that $u \in C^{\sigma+\alpha}(B_1) \cap C(\mathbb{R}^n)$ is a solution to

$$\begin{cases} M_{\mathcal{L}_0}^+ u \geq -C_0 & \text{in } B_1 \\ M_{\mathcal{L}_0}^+ u \leq C_0 & \text{in } B_1 \\ u = g & \text{in } \mathbb{R}^n \setminus B_1, \end{cases} \quad (4.12)$$

with $g \in C^\alpha(\mathbb{R}^n \setminus B_1)$. Then, $u \in C^\alpha(\mathbb{R}^n)$ with the estimate

$$\|u\|_{C^\alpha(\mathbb{R}^n)} \leq C(\|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_0), \quad (4.13)$$

where C depends only on $n, \sigma, \lambda, \Lambda$, and α .

Remark 4.4. The only “novelty” of the previous proposition with respect to the results in [4] is that there is no loss in the exponent: from a C^α exterior data we obtain C^α regularity up to the boundary (the same α). Note that in the proposition γ is small and $\alpha < \gamma$. Even for a linear translation invariant equations such a result is not true for all α . Indeed, even for the equation $\Delta u = 0$ in B_1 , it is well-known that Lipschitz boundary data may lead to a non-Lipschitz harmonic extension. The exponents $1, 2, 3, \dots$ (Lipschitz, $C^{1,1}$, $C^{2,1}$, ...) are in some sense critical for the boundary regularity of Δ because there exist harmonic polynomials that are degree $1, 2, 3, \dots$ and that solve $\Delta u = 0$ in \mathbb{R}_+^n and $u = 0$ on $\{x_n = 0\}$. Related to this, it is worth it to point out that a small modification of the proof of Proposition 4.3 shows that solutions to $(-\Delta)^s = 0$ in B_1 with C^α exterior data are C^α up to the boundary whenever $\alpha < s$. However, we do not expect the result to be true for $\alpha = s$. Again, the criticality of the exponent s comes from the fact that $(x_n)_+^s$ is a solution to the fractional Laplacian equation in the half space.

Proposition 4.3 will follow from an easy blow-up and compactness argument and from the following Liouville type result

Lemma 4.5. *Let $\sigma, \lambda, \Lambda, \gamma, \alpha$, as in the statement of Proposition 4.3.*

Assume that $u \in C(\mathbb{R}^n)$ is a viscosity solution to

$$\begin{cases} M_{\mathcal{L}_0}^+ u \geq 0 & \text{in } H \\ M_{\mathcal{L}_0}^+ u \leq 0 & \text{in } H \\ u = 0 & \text{in } \mathbb{R}^n \setminus H, \end{cases}$$

where H is the whole \mathbb{R}^n or some half space, and assume that u satisfies the growth control

$$\|u\|_{L^\infty(B_R)} \leq CR^\gamma$$

for all $R \geq 1$.

Then, u is constant ($u \equiv 0$ when $H \neq \mathbb{R}^n$).

Proof. Let $\rho \geq 1$ and $\bar{u}(x) = \rho^{-\alpha} u(\rho x)$. Letting $\bar{H} = H/\rho$, we have that \bar{u} solves

$$\begin{cases} M_{\mathcal{L}_0}^+ \bar{u} \geq 0 & \text{in } \bar{H} \cap B_1 \\ M_{\mathcal{L}_0}^+ \bar{u} \leq 0 & \text{in } \bar{H} \cap B_1 \\ \bar{u} = 0 & \text{in } B_1 \setminus \bar{H}. \end{cases}$$

In addition \bar{u} satisfies the growth control $\|\bar{u}\|_{L^\infty(B_R)} = \|\rho^{-\alpha} u\|_{L^\infty(B_{\rho R})} \leq CR^\alpha$. Thus, in particular $|u| \leq C$ in B_1 and $\int_{\mathbb{R}^n} |u(y)| (1 + |y|)^{-n-\sigma} dy \leq C$.

Therefore it follows, using the interior and boundary regularity results from [3] and [4], that

$$\|\bar{u}\|_{C^\gamma(B_{1/16})} \leq C, \quad (4.14)$$

for some small γ depending only on n, σ, λ , and Λ .

Let us next give the details of the proof of (4.14). There are only two nontrivial cases: that \bar{H} contains $B_{1/8}$, or that $\partial\bar{H}$ has nonempty intersection with $B_{1/8}$. Otherwise $\bar{H} \cap B_{1/4} = \emptyset$ and (4.14) is trivial since $u \equiv 0$ in $B_{1/8}$.

In the first case ($B_{1/8} \subset \bar{H}$), (4.14) follows from the interior estimates in [4].

In the second case, there will be some point z in the intersection $\partial\bar{H} \cap B_{1/8}$, and \bar{u} solves an equation in half of $B_{1/2}(z)$ and vanishes in the complementary half ball. Then, a barrier argument shows that, for some small $p > 0$,

$$|\bar{u}| \leq C \text{dist}(x, \mathbb{R}^n \setminus \bar{H})^p \quad \text{in } B_{1/4}(z). \quad (4.15)$$

Indeed, the function $\psi(x) = \text{dist}(x, B_{1/10})^p$ is, for p small enough, a supersolution in the annulus $B_{1/10+\epsilon} \setminus B_{1/10}$, for some $\epsilon > 0$. Namely, it satisfies $M_{\mathcal{L}_0}^+ \psi \leq 0$ there —see for instance [4, Lemma 3.3]. Using translates of $C\psi$ (respectively $-C\psi$) as upper (lower) barrier we readily show (4.15). Combining it with the interior estimates —this is standard, see for instance the proof of Theorem 3.3 in [4]— we obtain

$$\|\bar{u}\|_{C^\gamma(B_{1/4}(z))} \leq C$$

for some $\gamma > 0$ (smaller than p and than the exponent of interior regularity). Then (4.14) follows since clearly $B_{1/16} \subset B_{1/4}(z)$ —recall that $z \in B_{1/8}$.

Finally we scale back (4.14) from \bar{u} to u and we obtain that, for all $\rho \geq 1$,

$$[u]_{C^\gamma(B_{\rho/16})} \leq C\rho^{\alpha-\gamma}.$$

Sending $\rho \nearrow +\infty$ we obtain that $[u]_{C^\gamma(\mathbb{R}^n)} = 0$ and thus u is constant. \square

Let us now give the

Proof of Proposition 4.3. Since u is a solution of (4.12) then by [4, Theorem 3.3] that $\|u\|$ satisfies the estimate

$$\|u\|_{C^{\alpha'}(B_1)} \leq C(\|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_0) \quad (4.16)$$

for some $\alpha' > 0$ and C depending only on $n, \sigma, \lambda, \Lambda$. Note that although Theorem 3.3 in [4] is stated with a general modulus of continuity, a inspection of its proof shows that a Hölder modulus of continuity for the exterior datum leads to another (worse) Hölder modulus of continuity up to the boundary.

By homogeneity we may always assume that $\|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_0 = 1$.

We want to show that the previous estimate (4.16) holds also with α' replaced by α , provided that $\alpha \in (0, \gamma)$, where γ is the exponent from Lemma 4.5. That is, we want to establish (4.13). The proof is by contradiction.

Similarly as in the proof of Proposition 3.1, if the estimate (4.13) is false then, for each integer $k \geq 0$, there exists g_k , $C_{0,k}$, and u_k , satisfying (4.12) —with u and g replaced by u_k and g_k respectively— such that

$$\|u_k\|_{C^\alpha(B_1)} \geq k,$$

while $\|g_k\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_{0,k} = 1$.

Using Lemma 3.2 we then have

$$\sup_k \sup_{z \in B_1} \sup_{r > 0} r^{\alpha' - \alpha} [u_k]_{C^{\alpha'}(B_r(z))} = +\infty, \quad (4.17)$$

Next we define

$$\theta(r) := \sup_k \sup_{z \in B_{1/2}} \sup_{r' > r} (r')^{\alpha' - \alpha} [u_k]_{C^{\alpha'}(B_{r'}(z))}.$$

Note that θ is monotone nonincreasing and $\theta(r) < +\infty$ for $r > 0$ since we are assuming that $\|g_k\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_{0,k} = 1$ and hence by (4.16) we have $\|u_k\|_{C^{\alpha'}(\mathbb{R}^n)} \leq C$. In addition, by (4.17) we have $\theta(r) \nearrow +\infty$ as $r \searrow 0$.

As in the proof of Proposition 3.1, there are sequences $r_m \searrow 0$, k_m , and $z_m \rightarrow z \in \overline{B_{1/2}}$, for which

$$(r'_m)^{\alpha' - \alpha} [u_{k_m}]_{C^{\alpha'}(B_{r'_m}(z_m))} \geq \frac{1}{2} \theta(r'_m). \quad (4.18)$$

We then consider the blow-up sequence

$$v_m(x) = \frac{u_{k_m}(z_m + r'_m x) - u_{k_m}(0)}{(r'_m)^\alpha \theta(r'_m)}.$$

Note also that (4.18) is equivalent to the following inequality for all $m \geq 1$:

$$[v_m]_{C^{\alpha'}(B_1)} \geq 1/2, \quad (4.19)$$

Similarly as in the proof of Proposition 3.1 we obtain

$$[v_m]_{C^{\alpha'}(B_R)} \leq C R^{\alpha - \alpha'} \quad \text{for all } R \geq 1. \quad (4.20)$$

and

$$\|v_m\|_{L^\infty(B_R)} \leq C R^\alpha \quad \text{for all } R \geq 1. \quad (4.21)$$

As in the proof of Proposition (3.1), by further rescaling v_m if necessary, we may assume that in addition to (4.19) the following holds

$$\text{osc}_{B_1} v_m \geq 1/4. \quad (4.22)$$

Using (4.20), (4.21) and the stability results for viscosity supersolutions and subsolutions [4, Lemma 4.3] we obtain that a subsequence of v_m converges locally uniformly in \mathbb{R}^n to a function v satisfying the assumptions of Lemma 4.5. Hence, v is constant. Since $v_m(0) = 0$ for all m we must have $v \equiv 0$, but then we reach a contradiction passing (4.22) to the limit. \square

We finally give the

Proof of Theorem 1.3. Let u^ϵ be the solution to (4.5), whose existence is guaranteed by Proposition 4.2.

Since I is an operator of the form (1.7) satisfying (1.8)-(1.9)-(1.10) then so is the regularized operator I^ϵ up to replacing λ, Λ by $\lambda/C, C\Lambda$ —see Remark 4.1. Note that $M_{\mathcal{L}_0}^+ u^\epsilon \geq -\sup_{a \in A} \|c_a\|_{L^\infty(B_1)} \geq -C_0$ in B_1 and similarly $M^- u^\epsilon \leq C_0$ in B_1 . Then, Theorem 3.3 in [4] provides with a modulus of continuity in $\overline{B_1}$ for u^ϵ — this modulus of continuity depends on the modulus of continuity of $g, n, \sigma, \lambda, \Lambda$, and C_0 , but not on ϵ . Therefore, using the Ascoli-Arzelà theorem, is a sequence $\epsilon_m \searrow 0$ and a function $u \in C^0(\overline{B_1})$ such that $u^{\epsilon_m} \rightarrow u$ uniformly in $\overline{B_1}$. Since $I^\epsilon \rightarrow I$ weakly as $\epsilon \searrow 0$, it follows from the “stability lemma” [4, Lemma 4.3] that the limiting function u is a viscosity solution of $I(u, x) = 0$ in B_1 that attains continuously the complement data g .

Let us prove that in both cases (a) and (b) the viscosity solution u belongs to $C^{\sigma+\alpha}(B_1)$ and hence it is a classical solution. For any $z \in B_1$ and $\rho > 0$ such that $B_\rho \subset B_1$ consider the rescaled function $\bar{u}^\epsilon(\bar{x}) = u^\epsilon(z + \rho\bar{x})$. Exactly as in the proof of Theorem (1.1), the function \bar{u} satisfies in B_1 the rescaled equation $\bar{I}^\epsilon(\bar{u}, \bar{x}) = 0$, where \bar{I}^ϵ is still of the form (1.7)-(1.10)-(1.11) with the same C_0, A_0 and ellipticity constants as I^ϵ .

In the case (a), using Proposition 4.3 we find that $u^\epsilon \in C^\alpha(\mathbb{R}^n)$ with

$$\|u^\epsilon\|_{C^\alpha(\mathbb{R}^n)} \leq C(\|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)} + C_0).$$

Therefore, since $\|\bar{u}^\epsilon\|_{C^\alpha(\mathbb{R}^n)} \leq \|u^\epsilon\|_{C^\alpha(\mathbb{R}^n)}$, applying Theorem 1.1 to the function $\bar{u}^\epsilon \in C^{\sigma+\alpha}(B_1)$ we obtain the estimate

$$\|\bar{u}^\epsilon\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)}). \quad (4.23)$$

Since $u^{\epsilon_m} \rightarrow u$ uniformly in B_1 , it follows that $\bar{u}^{\epsilon_m} \rightarrow \bar{u}$ uniformly in B_1 and thus, passing (4.23) to the limit we find

$$\|\bar{u}\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{C^\alpha(\mathbb{R}^n \setminus B_1)}).$$

This implies that u is $C^{\sigma+\alpha}$ in $B_{\rho/2}(z)$ — since $u(x) = \bar{u}(\frac{x-z}{\rho})$. Since all these balls $B_{\rho/2}(z)$ cover B_1 we have $u \in C^{\sigma+\alpha}(B_1)$. Moreover, when we take $z = 0$ and $\rho = 1$ we then have $\bar{u} \equiv u$ and we the previous estimate for \bar{u} yields the desired estimate for $\|u\|_{C^{\sigma+\alpha}(B_{1/2})}$.

In the case (b), using the trivial barriers we prove that

$$\|\bar{u}^\epsilon\|_{L^\infty(\mathbb{R}^n)} = \|u^\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq C(\|g\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + C_0).$$

Therefore, using Corollary 1.2 applied to the function \bar{u} we obtain that

$$\|\bar{u}^\epsilon\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{L^\infty(\mathbb{R}^n \setminus B_1)})$$

where C_0 is the constant from (1.10) and C depends only on $n, \sigma, \alpha, \lambda, \Lambda$, and A_0 . Again, this implies that $u \in C^{\sigma+\alpha}(B_1)$ and the estimate for $\|u\|_{C^{\sigma+\alpha}(B_{1/2})}$.

Finally, in both cases (a) and (b), after having proved the existence of a classical solution (attaining continuously the complement data), its uniqueness among the

class of viscosity solutions follows from the trivial comparison principle between a classical solution and a viscosity solution. \square

5. COUNTEREXAMPLES TO $C^{\sigma+\alpha}$ REGULARITY FOR MERELY BOUNDED COMPLEMENT DATA

In this section we find sequences u_m of solutions to equations with rough kernels in B_1 that satisfy $\|u_m\|_{L^\infty(\mathbb{R}^n)} \leq C$ and $\|u_m\|_{C^{\sigma+\alpha}(B_{1/2})} \nearrow \infty$ as $m \rightarrow \infty$ for all $\alpha > 0$. We consider the case of a linear equation and the case of a nonlinear convex equation involving the extremal operator $M_{\mathcal{L}_0}^+$. Such sequences can be regarded as counterexamples to a $C^{\sigma+\alpha}$ interior estimate for linear or convex equations with rough kernels with merely bounded complement data. These counterexamples are built here in dimension $n = 1$. Clearly, looking at these one-dimensional counterexamples as 1-D profiles in \mathbb{R}^n we will have counterexamples in every dimension.

We will need the following elementary

Claim 5.1. *Assume that some function u and $\alpha > 0$ it is*

$$\|u\|_{C^{\sigma+\alpha}(-1/2, 1/2)} \leq C_0$$

and

$$\|u\|_{C^\alpha(\mathbb{R})} \leq C_0.$$

Then, for all $L \in \mathcal{L}_0$ we have

$$\|Lu\|_{C^{\alpha'}(-1/4, 1/4)} \leq CC_0, \tag{5.1}$$

where $\alpha' > 0$ and C depend only on σ and ellipticity constants.

Proof. We have

$$|\delta_2 u(x_1, y) - \delta_2 u(x_2, y)| \leq \begin{cases} CC_0 |y|^{\sigma+\alpha} \\ C_0 |x_1 - x_2|^\alpha \\ CC_0 |y|^{\theta(\sigma+\alpha)} |x_1 - x_2|^{(1-\theta)\alpha}. \end{cases}$$

The third bound is obtained by “interpolating” the first and the second ones.

But then for all $x_1, x_2 \in (-1/4, 1/4)$

$$\begin{aligned} |Lu(x_1) - Lu(x_2)| &\leq C \int_{\mathbb{R}} \frac{|\delta_2 u(x_1, y) - \delta_2 u(x_2, y)|}{|y|^{n+\sigma}} dy \\ &\leq C \int_{-1}^1 \frac{|y|^{\theta(\sigma+\alpha)} |x_1 - x_2|^{(1-\theta)\alpha}}{|y|^{1+\sigma}} dy + \int_{\mathbb{R} \setminus (-1, 1)} \frac{|x_1 - x_2|^\alpha}{|y|^{1+\sigma}} dy \\ &\leq C |x_1 - x_2|^{\alpha'}, \end{aligned}$$

where we have taken $\theta < 1$ very close to 1 such that $\theta(\sigma + \alpha) > \sigma$ and $\alpha' = (1 - \theta)\alpha$. \square

We note that with the same assumptions of the Claim it is possible (and not difficult) to prove that (5.1) holds for $\alpha' = \alpha$ but the previous rough version will suffice for our purposes.

5.1. Linear equations with rough kernels. In \mathbb{R} , for every integer $m \geq 1$ consider the function u_m that solves

$$\begin{cases} L_m u_m = 0 & \text{in } (-1, 1) \\ u_m = 0 & \text{in } [-2, -1] \cup [1, 2] \\ \text{sign} \sin(m\pi x) & \text{in } (-\infty, -2] \cup [2, \infty), \end{cases}$$

where L_m is defined by

$$L_m v = \int_{\mathbb{R}} \delta^2 v(x, y) K_m(y) dy$$

for

$$K_m(y) = \begin{cases} |y|^{-1-\sigma} & \text{in } (-1, 1) \\ (2 + \text{sign} \cos(m\pi y)) |y|^{-1-\sigma} & \text{in } (-\infty, -1) \cup (1, +\infty). \end{cases}$$

Next we use that for $p > 0$ small enough the function $\psi(x) = \text{dist}(x, [-1/4, 1/4])^p$ is a supersolution in $(-1/4 - \epsilon, -1/4) \cup (1/4, 1/4 + \epsilon)$, for some $\epsilon > 0$. Namely, it satisfies $M_{\mathcal{L}_0}^+ \psi \leq 0$ there —see [4]. Since $u_m \equiv 0$ in $(-2, -1) \cup (1, 2)$, by using translates of ψ (respectively $-\psi$) as upper (lower) barrier we prove that

$$|u_m| \leq C \text{dist}(x, (-\infty, -1] \cup [1, \infty))^p \quad \text{in } (-1, 1).$$

Combining this with known interior estimates we obtain, for small enough $\alpha > 0$,

$$\|u_m\|_{C^\alpha([-1, 1])} \leq C, \quad \text{for all } m.$$

Finally let us show that it is impossible that $\|u_m\|_{C^{\sigma+\alpha}(-1/2, 1/2)} \leq C$ with $\alpha > 0$ and C independent of m . Let us write $u_m = u_m^{(1)} + u_m^{(2)}$, where $u_m^{(1)} = u_m \chi_{(-1, 1)}$ and

$$u_m^{(2)} = \begin{cases} 0 & \text{in } (-2, 2) \\ \text{sign} \sin(m\pi x) & \text{in } (-\infty, -2] \cup [2, \infty). \end{cases}$$

We would then have

$$\|u_m^{(1)}\|_{C^{\sigma+\alpha}(-1/2, 1/2)} + \|u_m^{(1)}\|_{C^\alpha(\mathbb{R})} \leq C.$$

Thus using Claim 5.1 we would obtain

$$\|L_m u_m^{(1)}\|_{C^{\alpha'}(-1/4, 1/4)} \leq C,$$

with C independent of m .

Next, on the other hand

$$L_m u_m^{(2)}(0) = 0 \quad \text{by odd symmetry of } u_m^{(2)}.$$

Let us now compute $L_m u_m^{(2)}\left(\frac{1}{2m}\right)$. We see that for $|y| > 2 + \frac{1}{2m}$ we have

$$\begin{aligned} \delta^2 u_m^{(2)}\left(\frac{1}{2m}, y\right) &= \frac{1}{2} \left\{ \text{sign} \sin\left(\frac{\pi}{2} + m\pi y\right) + \text{sign} \sin\left(\frac{\pi}{2} - m\pi y\right) - 2u_m^{(2)}\left(\frac{1}{2m}\right) \right\} \\ &= \text{sign} \cos(m\pi y) - 0. \end{aligned}$$

We thus obtain

$$\begin{aligned} L_m u_m^{(2)}\left(\frac{1}{2m}\right) &= \int_{\mathbb{R} \setminus (-2, 2)} \text{sign} \cos(m\pi y) \frac{2 + \text{sign} \cos(m\pi y)}{|y|^{1+\sigma}} dy + O(1/m) \\ &= c + 2 \int_{\mathbb{R} \setminus (-2, 2)} \frac{\text{sign} \cos(m\pi y)}{|y|^{1+\sigma}} dy + O(1/m) \\ &= c + o(1) \quad \text{as } m \nearrow \infty, \end{aligned}$$

where $c = \int_{\mathbb{R} \setminus (-2, 2)} |y|^{-1-\sigma} dy > 0$.

Therefore,

$$\begin{aligned} 0 &= L_m u_m\left(\frac{1}{2m}\right) - L_m u_m(0) \\ &\geq L_m u_m^{(2)}\left(\frac{1}{2m}\right) - L_m u_m^{(2)}(0) - |L_m u_m^{(1)}\left(\frac{1}{2m}\right) - L_m u_m^{(1)}(0)| \\ &\geq c - o(1) - C\left(\frac{1}{2m}\right)^{\alpha'} \quad \text{as } m \nearrow \infty; \end{aligned}$$

a contradiction.

5.2. Nonlinear convex equation with the $M_{\mathcal{L}_0}^+$. This is a variation of the previous example. In \mathbb{R} , for every $m \geq 1$ consider the function u_m that solves

$$\begin{cases} M_{\mathcal{L}_0}^+ u_m = 0 & \text{in } (-1, 1) \\ u_m = 0 & \text{in } [-2, -1] \cup [1, 2] \\ \text{sign} \sin(m\pi x) & \text{in } (-\infty, -2] \cup [2, \infty). \end{cases}$$

Since $-1 \leq u_m \leq 1$ in \mathbb{R} but $|\{u_m < 0\} \cap (-5, 5)| \geq 1$ and $M_{\mathcal{L}_0}^+ u = 0$ in $(-1, 1)$, it will be

$$1 \leq u_m \leq 1 - \tau \quad \text{in } [-1/2, 1/2] \tag{5.2}$$

for some $\tau > 0$ depending only on σ and ellipticity constants.

In addition, we use as in the previous subsection that for $p > 0$ small enough the function $\psi(x) = \text{dist}(x, [-1/4, 1/4])^p$ is a supersolution in $(-1/4 - \epsilon, -1/4) \cup (1/4, 1/4 + \epsilon)$, for some $\epsilon > 0$. Namely, it satisfies $M_{\mathcal{L}_0}^+ \psi \leq 0$ there. Since $u_m \equiv 0$ in $(-2, -1) \cup (1, 2)$, by using translates of ψ (respectively $-\psi$) as upper (lower) barrier we prove that

$$|u_m| \leq C \text{dist}(x, (-\infty, -1] \cup [1, \infty))^p \quad \text{in } (-1, 1).$$

Combining this with known interior estimates we obtain, for small enough $\alpha > 0$,

$$\|u_m\|_{C^\alpha([-2, 2])} = \|u_m\|_{C^\alpha([-1, 1])} \leq C, \quad \text{for all } m. \tag{5.3}$$

Next we use that for $|x| > 2$ we have $u(x) = \text{sign} \sin(m\pi x)$, which is odd, we obtain

$$\delta^2 u_m(0, y) = -u_m(0) \leq 0 \quad \text{for } |y| > 2. \quad (5.4)$$

The fact that $u_m(0) \geq 0$ can be easily deduced by observing that for all $L \in \mathcal{L}_0$ the solution to the linear equation $Lw = 0$ in $(-1, 1)$ with the same boundary data as u_m satisfies $w(0) = 0$ (by odd symmetry), and w it is a subsolution to our equation since $M_{\mathcal{L}_0}^+ w \geq Lw = 0$.

Let us denote

$$b_m(y) = \Lambda(\delta^2 u_m(0, y))^+ + \lambda(\delta^2 u_m(0, y))^- ,$$

that is,

$$M_{\mathcal{L}_0}^+ u_m(0) = \int_{\mathbb{R}} \delta^2 u(0, y) \frac{b_m(y)}{|y|^{n+\sigma}} dy. \quad (5.5)$$

Hence, by (5.4),

$$b_m(y) \equiv \lambda \quad \text{for } |y| > 2. \quad (5.6)$$

Instead, at $x = \frac{1}{2m}$ we have

$$\delta^2 u_m\left(\frac{1}{2m}, y\right) = \text{sign} \cos(m\pi y) - u_m\left(\frac{1}{2m}\right) \quad \text{for } y \in \mathbb{R} \setminus \left(-2 - \frac{1}{2m}, 2 - \frac{1}{2m}\right). \quad (5.7)$$

Hence if we let

$$\tilde{b}_m(y) = \Lambda\left(\delta^2 u_m\left(\frac{1}{2m}, y\right)\right)^+ + \lambda\left(\delta^2 u_m\left(\frac{1}{2m}, y\right)\right)^- ,$$

that is,

$$M_{\mathcal{L}_0}^+ u_m\left(\frac{1}{2m}\right) = \int_{\mathbb{R}} \delta^2 u_m\left(\frac{1}{2m}, y\right) \frac{\tilde{b}_m(y)}{|y|^{n+\sigma}} dy. \quad (5.8)$$

we then have

$$\tilde{b}_m(y) = \lambda + \frac{\Lambda - \lambda}{2}(1 + \text{sign} \cos(m\pi y)) \quad \text{for } y \in \mathbb{R} \setminus \left(-2 - \frac{1}{2m}, 2 - \frac{1}{2m}\right). \quad (5.9)$$

For all $\gamma \in (0, 1)$ and for all m using that $u_m(0) \in [0, 1 - \tau]$ by (5.2) we obtain

$$\int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} (\text{sign} \cos(m\pi y) - u_m(0)) \frac{\frac{\Lambda-\lambda}{2}(1 + \text{sign} \cos(m\pi y))}{|y|^{n+\sigma}} dy \geq 2c_1 > 0$$

where c_1 is independent on m —like τ . Therefore, for all m large enough so that $\left| \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} \text{sign} \cos(m\pi y) \frac{\lambda}{|y|^{n+\sigma}} dy \right| \leq c_1$ we have

$$\begin{aligned} & \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} (\text{sign} \cos(m\pi y) - u_m(0)) \frac{\lambda + \frac{\Lambda-\lambda}{2}(1 + \text{sign} \cos(m\pi y))}{|y|^{n+\sigma}} dy \geq \\ & \geq c_1 - \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} u_m(0) \frac{\lambda}{|y|^{n+\sigma}} dy. \end{aligned} \quad (5.10)$$

Next, from (5.10), using (5.4), (5.6), (5.7), and (5.9) we obtain (for m large enough, in particular $\frac{1}{2m} < \gamma$)

$$\begin{aligned} & \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} \delta^2 u\left(\frac{1}{2m}, y\right) \frac{\tilde{b}(y)}{|y|^{n+\sigma}} dy - \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} \delta^2 u(0, y) \frac{b(y)}{|y|^{n+\sigma}} dy \geq \\ & \geq c_1 - \left| u_m\left(\frac{1}{2m}\right) - u_m(0) \right| \int_{\mathbb{R} \setminus (-2-\gamma, 2+\gamma)} \frac{\lambda + \frac{\Lambda-\lambda}{2}(1 + \text{sign } \cos(m\pi y))}{|y|^{n+\sigma}} dy \quad (5.11) \\ & \geq c_1 - C \left| \frac{1}{2m} - 0 \right|^\alpha. \end{aligned}$$

In the last inequality we have used (5.3).

Let us show that it is impossible that $\|u_m\|_{C^{\sigma+\alpha}(-1/2, 1/2)} \leq C$, with $\alpha > 0$ and C independent of m .

To reach a contradiction let us show that the kernels $b_m(y)|y|^{-n-\sigma}$ and $\tilde{b}_m(y)|y|^{-n-\sigma}$ would “perform similarly” when integrated only in $(-2, 2)$ against $\delta^2 u_m$ at the points 0 and $\frac{1}{2m}$. More precisely, let us prove the bound

$$\left| \int_{(-2+\gamma, 2-\gamma)} \delta^2 u(0, y) \frac{b(y)}{|y|^{n+\sigma}} dy - \int_{(-2+\gamma, 2-\gamma)} \delta^2 u\left(\frac{1}{2m}, y\right) \frac{\tilde{b}(y)}{|y|^{n+\sigma}} dy \right| \leq C(1/m)^{\alpha'}, \quad (5.12)$$

for some $\alpha' \in (0, \alpha)$. To prove (5.12) we use that for $1/m < \gamma$ and $|y| < 2 - \gamma$ we would have

$$\left| \delta^2 u_m(0, y) - \delta^2 u_m\left(\frac{1}{2m}, y\right) \right| \leq \begin{cases} C|y|^{\sigma+\alpha} \\ C\left|0 - \frac{1}{2m}\right|^\alpha. \end{cases}$$

The first bound is obtained from the assumption $\|u_m\|_{C^{\sigma+\alpha}(-1/2, 1/2)} \leq C$ and the second from (5.3). Hence, “interpolating” the two bounds we obtain

$$\left| \delta^2 u_m(0, y) - \delta^2 u_m\left(\frac{1}{2m}, y\right) \right| \leq C_1 |y|^{\theta(\sigma+\alpha)} \left| \frac{1}{2m} \right|^{(1-\theta)\alpha} = C_1 |y|^{\sigma+\alpha'} \left| \frac{1}{2m} \right|^{\alpha'},$$

where we have taken $\theta \in (0, 1)$ close to 1 and $\alpha' > 0$ so that $\theta(\sigma + \alpha) = \sigma + \alpha'$ and $(1 - \theta)\alpha = \alpha'$.

Therefore if we split the interval $(-2 + \gamma, 2 - \gamma)$ into the three disjoint subsets

$$\mathbf{A} = \left\{ y \in (-2 + \gamma, 2 - \gamma) : \delta^2 u_m(0, y) \geq 2C_1 |y|^{\sigma+\alpha'} \left| \frac{1}{2m} \right|^{\alpha'} \right\},$$

$$\mathbf{B} = \left\{ y \in (-2 + \gamma, 2 - \gamma) : \delta^2 u_m(0, y) \leq -2C_1 |y|^{\sigma+\alpha'} \left| \frac{1}{2m} \right|^{\alpha'} \right\},$$

and

$$\mathbf{C} = (-2 + \gamma, 2 - \gamma) \setminus (\mathbf{A} \cup \mathbf{B}).$$

Then it is $b = \tilde{b} = \Lambda$ in \mathbf{A} , $b = \tilde{b} = \lambda$ in \mathbf{B} and

$$\left| \delta^2 u_m(0, y) \right| + \left| \delta^2 u_m\left(\frac{1}{2m}, y\right) \right| \leq 4C_1 |y|^{\sigma+\alpha'} (1/m)^{\alpha'} \quad \text{in } \mathbf{C}.$$

This clearly implies (5.12).

Finally, using (5.5), (5.8), (5.11), and (5.12), we obtain

$$\begin{aligned} 0 &= M_{\mathcal{L}_0}^+ u_m \left(\frac{1}{2m} \right) - M_{\mathcal{L}_0}^+ u_m(0) \\ &\geq c_1 - C(1/m)^\alpha - C(1/m)^{\alpha'} - C\gamma, \end{aligned}$$

which yields to contradiction taking first $\gamma < c_1/C$ and then m large enough.

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